C^k Continuity of Subdivision Surfaces

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Abstract

Stationary subdivision is an important tool for generating smooth free-form surfaces for CAGD and computer graphics. One of the challenges in construction of subdivision schemes for arbitrary meshes is to guarantee that the limit surface will have smooth regular parameterization in a neighborhood of any point. First results in this direction were obtained only recently. In this paper we derive necessary and sufficient criteria for C^k-continuity that generalize and extend most known conditions. We create a general mathematical framework that can be used for analysis of more general types of schemes. Finally, we prove a degree estimate for C^k-continuous polynomial schemes generalizing an estimate of Reif [20] and give a practical sufficient condition for smoothness.

1 Introduction

The main application of subdivision in computer graphics and CAGD is generation of smooth or piecewise smooth surfaces.

Given an initial mesh, subdivision computes a sequence of refined meshes converging to a limit surface. The refined meshes are obtained by adding new vertices to the mesh and connecting them with old vertices. The positions of new vertices are computed as functions of positions of the old vertices; the positions of old vertices in the refined mesh can be modified. To specify a subdivision scheme, we need to describe two rules: a topological rule for obtaining the graph of the refined mesh from the graph of the initial mesh (which edges to insert, which edges to remove) and a rule for computing the positions of new vertices and modifying positions of the old vertices.

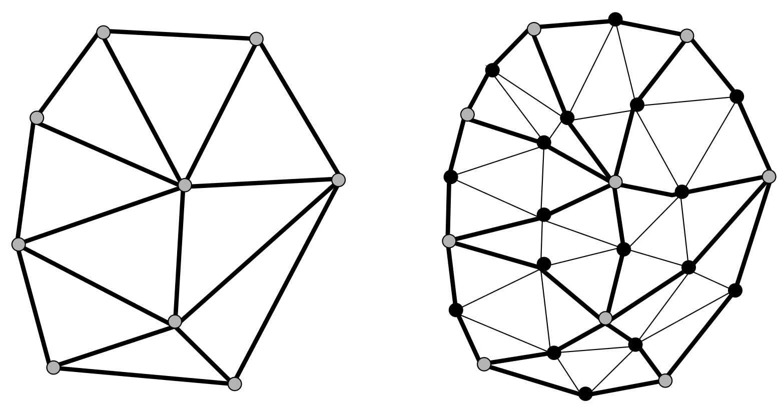

Figure 1: Insertion of new vertices for a triangular mesh.

The topological rule has primary importance; in fact, only several rules where ever used for construction of subdivision schemes for surfaces. In this paper we will consider the schemes that use probably the simplest possible rule that works on arbitrary triangular meshes: one new vertex is added for each old edge, all old edges are replaced by a pair of edges and the new vertices for each old face are connected (Figure 1).

This rule is general enough to allow for a variety of subdivision algorithms. An important property of this rule is that all new vertices, unless they are on the boundary of the mesh, have valence 6, i.e. have 6 edges connected to them. The new vertices on the boundary have valence four. The number of *extraordinary* vertices, which have valence other then 6 (other then 4 on the boundary), remains constant on all subdivision levels. For the type of schemes that we consider, this fact means that almost everywhere a small part of the mesh is in one-to-one correspondence with a piece of the regular three-directional grid (Figure 1). Powerful tools exist for analysis and construction of subdivision schemes on such grids (see [5]). Thus our task is reduced to analysis of the behavior of subdivision near the extraordinary vertices.

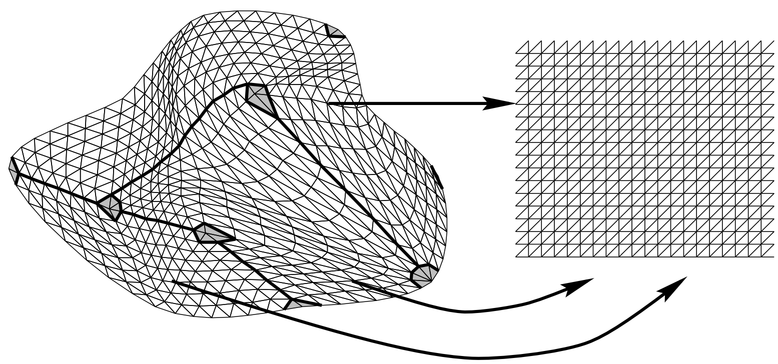

Figure 2: Each piece of the of the subdivided mesh, except for the shaded ones, can be mapped to the regular grid with a boundary.

The second rule that we need to specify a subdivision scheme, is the set of functions used to compute the positions of vertices on successive approximation levels. For the scheme to be practical, these functions should be as simple as possible, while allowing to generate a smooth limit surface. We will consider subdivision schemes for which all refinement functions are linear. Moreover, we will assume that the choice of function that is used to compute a value at a vertex depends only on the local topology of the mesh around this vertex, and the function itself depends only on a finite number of neighbors (*locality and compact support.*)

Linear and compactly supported functions can be computed very efficiently, which makes the schemes of this type particularly promising for computer graphics applications.

We will also assume that the schemes are *stationary*, which implies that the choice of the refinement functions does not depend on the subdivision level. Once a mesh is refined, we "forget" about the old mesh and base our choice of functions for the next subdivision step only on the topology of the current mesh. This restriction makes implementation highly efficient and also make the analysis of the schemes much simpler. Following the pattern that was established in the analysis of subdivision on regular grids [9], one may hope to extend this analysis to the case of non-stationary subdivision. As we will show in this paper this restriction is very significant. Although it is possible to obtain schemes of arbitrary smoothness, their support is likely to be large; it appears to be impossible to design stationary schemes minimizing a particular fairing functional; it is difficult to find a scheme which produces surfaces that are not only smooth but also reasonably fair.

The goal of this paper is to build a systematic theory of stationary subdivision surfaces on arbitrary meshes. We find necessary and sufficient conditions for C^k-continuity of subdivision schemes. These conditions can be used to evaluate smoothness of particular schemes; more importantly, they provide us with a more explicit description of the whole class of C^k-continuous schemes; it is our hope that such description can be used for finding schemes in the class that are optimal in other senses, most importantly, schemes that produce surfaces with improved fairness.

Our analysis of stationary subdivision around extraordinary vertices builds on the ideas from the work of Warren [24], Reif [19, 18, 20] and Cavaretta, Dahmen and Miccelli [5].

Main results. Our main results include

- necessary and sufficient conditions for geometric continuity (existence of limit normals);
- necessary and sufficient conditions for C^k-continuity of subdivision schemes at extraordinary vertices under some assumptions (Section 5);
- a practical sufficient criterion for smoothness of a subdivision scheme around an extraordinary vertex (Section 7);
- an extension of a result by Reif [20] on the degree estimate of C^2-continuous piecewise polynomial subdivision to the case of C^k-continuity.
- a formalism for description of subdivision schemes on arbitrary meshes in terms of graph neighborhoods (Section 2).

The necessary and sufficient criteria for C^k-continuity also give some clues for construction of C^k subdivision schemes, in particular, schemes on tagged meshes, where non-standard behavior is desired for tagged vertices, edges or faces.

The sufficient conditions of Section 7 does not require knowledge of the explicit formula for the limit surface, which makes it useful for analysis of interpolating subdivision schemes, such as the Butterfly scheme [10]. Moreover, the condition is "almost necessary", as its converse is necessarily true for schemes satisfying nondegeneracy conditions of Section 5.1.

In Section 2 we build a formal mechanism for describing arbitrary subdivision, which allows to define formally such crucial concepts as control sets (extension to the case of general subdivision of the idea of the set of control points of a spline patch).

The connection to the singularity theory briefly discussed in Section 6.3, suggests that some results from that area can be applied to study of subdivision surfaces, stationary as well as non-stationary.

Overview of the paper. In Section 2 we discuss the structure of the complexes generated by subdivision and prove some basic facts about dependencies of vertices generated by subdivision on the vertices on coarser levels.

Section 3 introduces topology on the complexes and defines convergence of subdivision. We prove that the any limit surface defined as a function on the initial mesh, can be decomposed into the linear combination of basis functions. In the same section we show that the analysis of any local and compactly supported stationary scheme on arbitrary complexes can be reduced to the analysis of that scheme on k-regular complexes, which have only one extraordinary vertex.

Section 4 introduces the *subdivision matrix.* It is shown that the limit functions generated by the eigenvectors of the matrix satisfy *scaling relations.* This section also states a convergence criterion for subdivision in terms of the eigenvalues of the subdivision matrix and describes decomposition of the limit surface in terms of *eigenbasis functions.*

Section 5 introduces our main results on C^k-continuity and provides a brief discussion. These results are based on properties of functions satisfying scaling relations, discussed in the next section.

Section 6 contains a discussion of the properties of the functions satisfying scaling relations.

First important results are contained in Section 5. We formulate criteria for C^k-continuity of a subdivision scheme in terms of a *characteristic map*, which is a map from the plane into the plane defined by the pair of dominant eigenvectors. We prove a necessary condition that should be imposed on the characteristic map for C^k-smoothness of the limit map. We show that injectivity of characteristic map formally is not a necessary condition for smoothness.

In Section 5.6 we describe our extension of Reif's degree estimate.

in Section 7 we formulate necessary and sufficient conditions relating injectivity of the characteristic map and its linear approximations.

Finally, Section 8 contains discussion of the results and future work.

In the beginning of each section we provide a short summary and informal discussion of the results.

1.1 Related Work

The subdivision literature is quite extensive, but until recently, surprisingly little was known about subdivision surfaces built on meshes with arbitrary topology. Already in the work of Doo and Sabin [6, 7], Catmull and Clark [4] attempts were made to analyze smoothness properties of the subdivision surfaces around extraordinary vertices. A more systematic approach was taken by Ball and Storry [3], who established conditions for geometric smoothness of Catmull-Clark subdivision. A similar analysis was performed by Loop [15].

Most recently, important results were obtained by Reif [19, 18, 20]. In [19] Reif points out that geometric smoothness does not adequately reflect the intuitive idea of smoothness and establishes sufficient conditions for a stronger notion of smoothness, which requires existence of a local regular parameterization. An important concept of a characteristic map is introduced. [20] demonstrates that polynomial patches of order 6 are required to achieve C^2-continuity of a surface — an important result showing limitations of stationary subdivision. The result requires establishing a necessary condition for C^2-continuity in a special case. In [18] asymmetric schemes are considered and somewhat more general sufficient conditions are proposed. Our work extends and generalizes the work of Reif. Another important source of ideas for this work was the manuscript by Warren [24], where he noted the importance of scaling relations for understanding behavior of subdivision schemes around extraordinary vertices.

Reif's results were used to analyze C^1-continuity of subdivision schemes by Habib and Warren [11], and Schweitzer [21].

This work fills several gaps in the existing literature. There was no rigorous framework for study of subdivision surfaces rather then of particular parameterizations of these surfaces. For smoothness of a surface, *existence* of a smooth parameterization is sufficient, rather then smoothness of any particular parameterization associated with the construction of the surface.

Many previous results implicitly or explicitly assumed that the limit surface of subdivision is composed of polynomial patches. Analysis of subdivision in the general case requires generalization of such notions as parametric domain and control points.

Our initial formalization is partly based on the theory developed for regular subdivision in Cavaretta, Dahmen and Micchelli [5].

Next, our goal was to find not only sufficient but necessary conditions for smoothness. Rather then requiring uniqueness of a characteristic map, we derive it from geometric smoothness and stability. We prove necessary conditions on eigenvalues of subdivision matrix that generalize all known necessary conditions and allow us to drop various ad hoc assumptions on largest eigenvalues that are typically made.

We establish necessary and sufficient conditions for C^k continuity and make them explicit for a broad class of subdivision schemes. These conditions allow us to prove a generalization of Reif's degree estimate for polynomial subdivision schemes.

Reif's sufficient conditions for smoothness of subdivision schemes requires establishing injectivity and regularity of the characteristic map. In [11] this was achieved using explicit representation of the surface with polynomial patches. Techniques developed in Schweitzer [21] are more subtle, but still rely on closed-form expressions. These approaches cannot be used to analyze schemes that do not have explicit formulas for parameterizations in the regular case. We propose a simple criterion for injectivity and regularity of a characteristic map using linear approximations. In a forthcoming paper we will use this method to establish C^1-continuity of an interpolating scheme.

When this work was complete, we learned about an earlier paper by Prautzsch [16], where C^k-continuity of surfaces was considered and sufficient conditions that form an important subset of our conditions were obtained.

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2 Subdivision of Abstract Complexes

In this section we define subdivision schemes on abstract simplicial complexes; abstract complexes is simply a class of graphs with values attached to the vertices — no topology or metrics are defined. Starting with abstract complexes allows us to separate the discrete aspects of the subdivision process from the continuous aspects.

The only non-trivial restriction that we impose on the complex is that the triangles of the complex adjacent to a vertex should form one ring. It is easy to see that any mesh can be converted to this type by replicating the vertex which does not satisfy this restriction.

We define a formal language for describing local parts of an abstract complex (neighborhoods) and derive their elementary properties.

Subdivision schemes that we consider in this paper are *compactly supported*, *locally defined* and *stationary.* Informally, compact support means that only a finite number of vertices is used to compute the value at a new vertex. Local definition means that coefficients used to compute the new value depend only on the structure of a finite part of the complex. Stationarity means that for two complexes of the same structure the same subdivision rules are used.

We show that if a scheme is compactly supported, then all new values that are computed as the result of subdivision of a triangle depend only on the values at the original vertices in a finite neighborhood of the triangle.

This neighborhood is called *control set* of the triangle and is similar to the set of control points of a spline patch. Similarly, a *localization set* is the neighborhood of the triangle whose topology influences the choice of functions for subdivision on all levels.

2.1 Definitions

Definition 2.1. *An abstract 2D simplicial complex $K = (V, E, F)$ is a set of vertices V, a set of abstract edges, which are unordered pairs of vertices, $E \subset \{(v, w) |\ v, w \in V\}$ and a set of abstract triangles which are unordered triples of vertices, $F \subset \{(u, v, w) |\ u, v, w \in V\}$, satisfying the conditions below.*

In this section we will drop the word "abstract" from the names "abstract triangle" and "abstract edge", as we don't need to distinguish them from other triangles and edges.

The first three conditions formalize the intuitive idea of the complex consisting of triangles that are glued together. The last two conditions ensure that the neighborhood of any vertex has simple structure.

1. All edges of a triangle are in E: if $(u, v, w) \in F$, then $(u, v), (v, w), (w, u) \in E$
2. No "dangling" edges: if $(v, w) \in E$, then there is $u \in V$ such that $(u, v, w) \in F$. If an edge is shared by two triangles it is called an *internal* edge. Otherwise, it is called a *boundary* edge.
3. No isolated vertices: If $v \in V$, there is $w \in V$ such that $(v, w) \in E$.
4. No more then two triangles share an edge: If $(v, w) \in E$, there are no more then two vertices u_1, u_2 such that $(u_1, v, w) \in F$ and $(u_2, v, w) \in F$.
5. Consider the set of all triangles containing the vertex v. Consider the set of all vertices of these triangles, v itself excluded: $N(v) = \{v_0, v_1, ... v_{k-1}\}$. We will assume that this set is finite and there is a permutation π such that $(v_{\pi(i)}, v_{\pi(i+1)}) \in E$, for any $i = 0 \ldots k-2$. If $(v_{\pi(k-1)}, v_{\pi(0)}) \notin E$ the vertex is called a *boundary vertex*; otherwise, the vertex is called an *internal* vertex. (Figure 2.1).

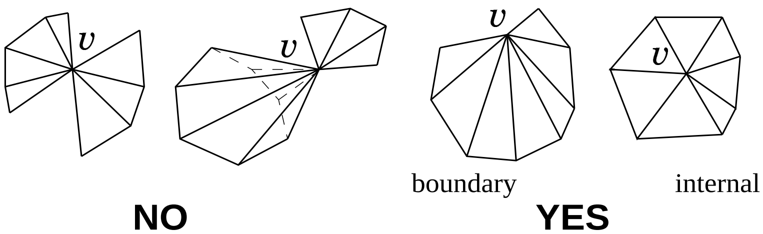

Figure 3: Left: Excluded configurations of triangles around a vertex. Right: These configurations are allowed.

Definition 2.2. *We will call two abstract simplicial complexes K_1 and K_2* isomorphic *if there is a one-to-one mapping $\rho : V_1 \to V_2$ such that $(v_i, v_j) \in E \Leftrightarrow (\rho(v_i), \rho(v_j)) \in E'$ and $(v_i, v_j, v_k) \in F \Leftrightarrow (\rho(v_i), \rho(v_j), \rho(v_k)) \in F'$. We use $\simeq$ to denote isomorphism.*

A *subcomplex* $K' = (V', E', F')$ of a complex K is a a simplicial complex that satisfies $V \subseteq V'$, $E \subseteq E'$, $F \subseteq F'$.

A simplicial complex is *closed* if all edges are internal. It is easy to show that all vertices are also internal in this case.

We call two vertices *adjacent* if they are connected by an edge.

k-regular complexes. Most of our constructions will use special types of complexes — *regular* and *k-regular*.

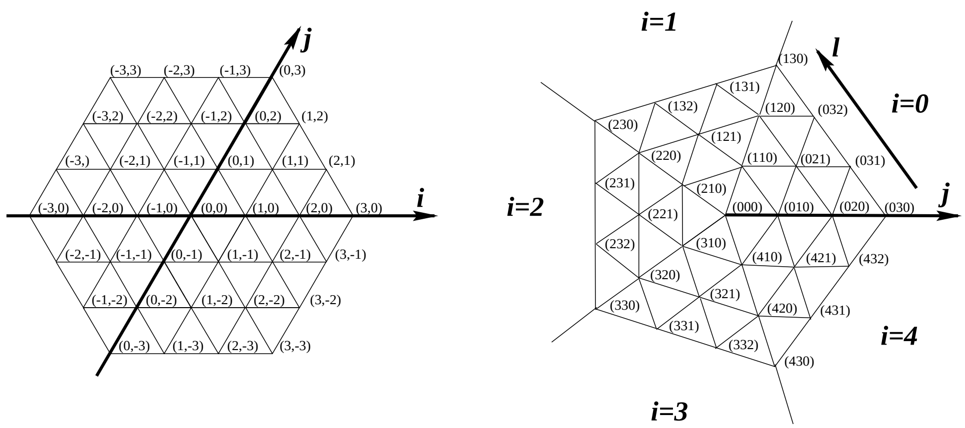

Figure 4: Left: enumeration of vertices of the regular complex. Right: enumeration of vertices of a k-regular complex for $k = 5$.

A k-regular complex $\mathcal{R}_k$ is shown in Figure 2.1. To specify the complex exactly, we need to enumerate its vertices and faces — all edges of each face are automatically included into the set of edges.

The set of vertices can be enumerated using three indices (i, j, l): the first index indicates the "sector", the other two define the position of the vertex in the sector. Indices satisfy $i = 0 \dots k-1$, $j > 0$, $l = 0 \dots j-1$. There is also a special central vertex $v_{0,0,0}$. A pair of faces is defined for each vertex $v_{i,j,l}$ except $v_{0,0,0}$ (Figure 2.1): for $l < j-1$ the triangles

$$\left(v_{i,j,l}, v_{i,j+1,l+1}, v_{i,j,l+1}\right), \ \left(v_{i,j,l}, v_{i,j-1,l}, v_{i,j,l+1}\right)$$

are in F; for $l = j-1$, the triangles

$$\left(v_{i,j,j-1}, v_{i,j+1,j}, v_{i+1,j,0}\right), \ \left(v_{i,j,j-1}, v_{i+1,j-1,0}, v_{i+1,j,0}\right)$$

are in F.

Index i is incremented modulo k, i.e. $v_{k,j,l} = v_{0,j,l}$.

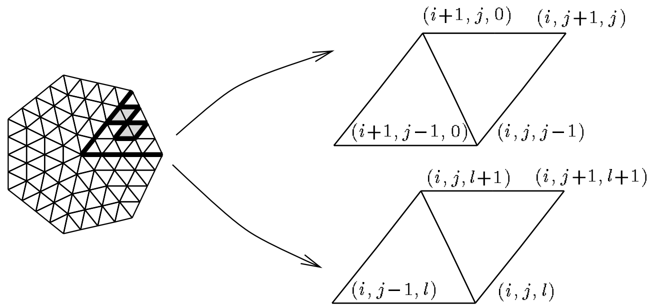

Figure 5: 2 faces corresponding to each vertex in a k-regular complex.

A k-regular complex with boundary $\mathcal{R}_k^B$ is defined in a similar way: the vertices of the complex are numbered using the same three-index scheme, but the last sector with $i = k - 1$ contains only vertices with $l = 0$.

The faces of the k-regular complex with boundary defined exactly in the same way, but i varies in the range $0 \ldots k - 2$.

If $k = 6$, there is a simpler way to enumerate the vertices of complex: they can be identified with the vertices of an integer grid on the plane. The vertex $v_{i,j}$ corresponds to the point (i, j) (Figure 2.1).

We will use this numbering for the 6-regular complex; we will call 6-regular complex simply regular.

Similar numbering can be introduced for the 4-regular complex with boundary: the vertices are $v_{i,j}$, where i is any integer, j is non-negative. We will call a 4-regular complex with boundary a regular complex with boundary.

2.2 Subdivision on Abstract Complexes

We start with a series of definitions formalizing the standard idea of subdivision.

Complex refinement. First we introduce *complex refinement* which describes how new vertices are added to a complex (this part of subdivision process was called “topological rule” in the introduction).

The procedure is an abstraction of the midpoint subdivision: insert a new vertex at the middle of each edge and connect the midpoints for each triangle. We call this procedure complex refinement.

let $D(K)$ be a new complex obtained in the following way: The new set of vertices $V' = V \cup V(E)$ where $V(E)$ is a set of vertices which has one vertex per edge of the complex K; let m_{vw} be the vertex in $V(E)$ corresponding to the edge (v, w).

The new set of edges E' is defined as $\{(v, m_{vw}) | (v, w) \in E\}$. For each old edge (v, w) there are two new edges (v, m_{vw}) and (w, m_{vw}) in E'.

The new set of triangles F' consists of four new triangles for each triangle in F: if $(u, v, w) \in F$, then (u, m_{uv}, m_{uw}), (v, m_{vw}, m_{uw}), (w, m_{uw}, m_{vw}) and (m_{uv}, m_{vw}, m_{uw}) are in E'. (Figure 2.2).

The new triangles are called *children* of the original triangle (u, v, w). The triangles of the complexes $D^j(K)$, $j = 0, 1 \ldots$ form a quadtree with no terminal nodes.

We use K^j as a notation for $D^j(K)$. The vertices of K^j are also vertices of K^{j+1}.

Notation for sets of vertices:

V_T^j is the set of vertices of all children of a triangle T in K^j.

V_T^∞ is the union of V_T^j for all j.

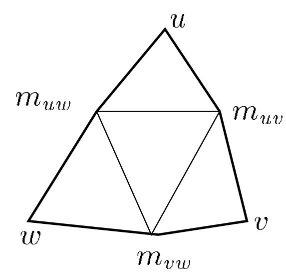

Figure 6: Insertion of new vertices for a triangle (u, v, w).

$V^j(A)$ is the union of V_T^j for all triangles T of a subcomplex A of K^j For $i < j$, $V^i(A)$ means $V^j(A) \cap V^i$ — the set of vertices of A that are vertices of K^i.

An isomorphism of complexes K_1 and K_2 can be extended to K_1^j and K_2^j:

$$\begin{aligned} \rho^j(v) &= \rho^{j-1}(v) \quad \text{for } v \in V_1^{j-1} \\ \rho^j(m_{uv}) &= m_{u'v'}, \quad \text{if } \rho^j(u) = u' \text{ and } \rho^j(v) = v'. \end{aligned} \tag{2.1}$$

where V_1 is the set of vertices of K_1.

A complex K is *self-similar*, if the complexes K and $D(K)$ are isomorphic. k-regular complexes and k-regular complexes with boundary are all self-similar. Recall that the set of vertices of $D(\mathcal{R}_k$ by definition consists of all vertices $v_{i,j,l}$ of $\mathcal{R}_k$ and vertices m_{uv} for all edges (u, v) of $\mathcal{R}_k$. For each vertex $v_{i,j,l}$ there are 3 new vertices in $D(\mathcal{R}_k)$, shown in Figure 2.2.

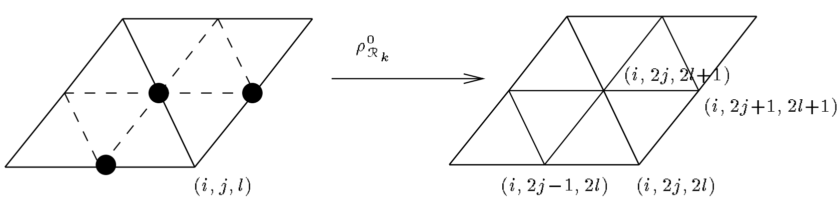

Figure 7: Refinement of k-regular complexes.

An isomorphism $\rho^0_{\mathcal{R}} : \ D(\mathcal{R}_k) \to \mathcal{R}_k$ is given by

$$\begin{aligned} \rho^0_{\mathcal{R}_k}(v_{i,j,l}) &= v_{2i,2j,2l} \\ \rho^0_{\mathcal{R}_k}(m(v_{i,j,l}, v_{i,j,l+1})) &= v_{i,2j,2l+1} \\ \rho^0_{\mathcal{R}_k}(m(v_{i,j,l}, v_{i,j,l+1})) &= v_{i,2j,2l+1} \\ \rho^0_{\mathcal{R}_k}(m(v_{i,j,l}, v_{i,j+1,l+1})) &= v_{i,2j+1,2l+1} \end{aligned} \tag{2.2}$$

Isomorphisms $\rho^j_{\mathcal{R}_k} : \mathcal{R}_k^{j+1} \to \mathcal{R}_k^j$ are derived from ρ using formulas (2.1).

Subdivision schemes. Next we attach values to the vertices of the complex. A set of values defined at each vertex of the complex can be regarded as a function on the set of vertices of this complex.

For a a fixed complex K, the linear vector space $\mathcal{P}(V, B)$ is defined as the space of functions defined on the set of vertices of the complex V with values in a vector space B, with operations defined in the natural way. We are interested in the cases when B is $\mathbf{R}^n$ or $\mathbf{C}^n$ for $n = 1, 2, 3$. If two complexes are isomorphic, an isomorphism $\rho : V_1 \to V_2$ induces an isomorphism ρ^* of linear spaces $\mathcal{P}(V_1, B)$ and $\mathcal{P}(V_2, B)$ via $\rho^*(p)(v) = p(\rho^{-1}(v))$.

$$\begin{tikzcd} V_1 \arrow[r, "\rho"] \arrow[d, "p\in\mathcal{P}(V_1)"'] & V_2 \arrow[dl, "\rho^*(p)\in\mathcal{P}(V_2)"] \\ B & \end{tikzcd}$$

Typically we omit B in $\mathcal{P}(V, B)$, as the range is often obvious from the context.

The most general definition of subdivision simply states that a subdivision scheme defines a way to compute values at finer subdivision levels from the values at the top level for any complex and any initial set of values.

Definition 2.3. *An subdivision scheme* $\mathcal{S}$ *is a map from the set of complexes to the sequences* $\{S^0[K], S^1[K], \dots\}$ *where* $S^j[K]$ *is a linear operator from* $\mathcal{P}(V^j)$ *to* $\mathcal{P}(V^{j+1})$*. We will call the operators* $S^j[K]$ *subdivision operators.*

This definition is too general to be useful. We require schemes to have a number of properties. These properties have two origins: most of them are motivated by practical considerations. One property (stationarity), while being practically useful, also makes theoretical analysis much easier.

Before formulating the properties of the schemes, we define *neighborhoods* and *stencils*.

Definition 2.4. *A* 1*-neighborhood* $N_1(W)$ *of a set of vertices vertices* $W \in V$ *is the subcomplex of* K *consisting of all triangles with at least one vertex in* W*, their edges and vertices. An* m*-neighborhood* $N_m(W)$ *is defined recursively as a 1-neighborhood of the* $m-1$*-neighborhood.*

$N_i^j(A)$ where A is a subcomplex of K^j denotes the i-neighborhood of the set of vertices of A.

Definition 2.5. *If a vertex* v *in* K^{j+1} *is fixed and* $p \in \mathcal{P}(K^j)$ *varies,*

$$S^j[K](p)(v) = S^j[K, v](p)$$

is a linear function from $\mathcal{P}(K^j)$ *to* B*. We call* $S^j[K, v]$ *a subdivision function.*

Another object that we use is the *stencil* or the *mask* of a subdivision scheme at a vertex v of a complex K^{j+1}, which is the support of the subdivision function at v.

Definition 2.6. *The stencil* $\mathrm{St}(K^j, v)$ *is the set of vertices of* K^j *such that* $S^j[K, v](p) = 0$ *if and only if* $p = 0$ *on* $\mathrm{St}(K^j, v)$.

Note that the vertices of the stencil are vertices of K^j but due to the inclusion $V^j \subset V^{j+1}$ they may be regarded also as vertices of K^{j+1}.

If the linear function $S^j[K, v]$ is written as a linear combination of $p(v)$,

$$S^j[K, v] = \sum a_v p(v)$$

then $\mathrm{St}(K^j, v)$ is the set of all v such that the coefficient a_v is not zero.

Example. Figure 2.2 shows the stencils of the Loop scheme for various vertices.

We consider subdivision schemes having the following properties:

- stationarity;
- compact support;
- local definition;
- affine invariance.

Stationarity. The subdivision operator S^j depends only on K^j up to an isomorphism, not on j or the particular choice of K. Suppose two complexes K_1^i and K_2^j are isomorphic, and $\rho : V_1^i \to V_2^j$ is the isomorphism; then $\rho \circ S^j[K_2](p) = \rho^* S^i[K_1](\rho^*(p))$, as it is shown in the diagram:

$$\begin{array}{ccc} \mathcal{P}(K_1^i) & \xrightarrow{S^i[K_1]} & \mathcal{P}(K_1^{i+1}) \\ \downarrow{\scriptstyle \rho^*} & & \downarrow{\scriptstyle \rho^*} \\ \mathcal{P}(K_2^j) & \xrightarrow{S^j[K_2]} & \mathcal{P}(K_2^{j+1}) \end{array}$$

For stationary schemes, we will use notation $S^j[K] = S[K^j]$.

It is also useful to consider *stationarity with respect to a set of isomorphisms,* In this case the relation between S^j and S^i should hold only if the isomorphism ρ comes from a given set. Stationary subdivision on tagged complexes can be described in using such sets. In our construction in this paper we will use stationarity only in a limited way, which makes our results useful for schemes that are stationary only with respect to certain sets of isomorphisms (see Section 8 for discussion).

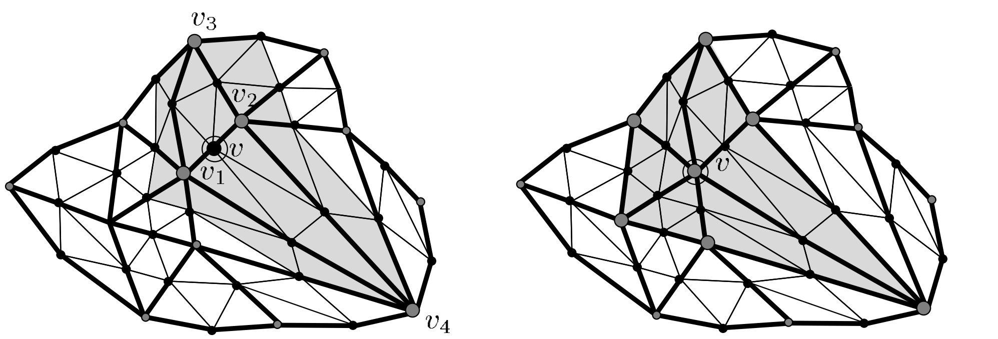

Figure 8: Locality and compact support for the Loop scheme, $L = M = 2$. The larger gray disks mark vertices of the stencils $S[K,v]$. Left: $N_M(v)$ for a new vertex v; Right: N_M for an old vertex v.

Compact support. The new value at any vertex depends only on the finite number of values on the coarser level in the neighborhood of the vertex.

Formally, suppose for any j and $v \in K^{j+1}$ there is a neighborhood $N_{M(v)}^{j+1}(v)$ containing the stencil $\mathrm{St}(K^j, v)$. If the set of all $M(v)$ for all K and j is bounded by a constant M, the scheme has compact support:

$$\text{for all } K, v \quad \mathrm{St}(K^j, v) \subset N_M^{j+1}(v)$$

We will call minimal M the *support size* for the subdivision scheme. Figure 2.2 shows stencils and neighborhoods N_m for the Loop scheme (see Example below).

Local definition. Assume that the scheme $\mathcal{S}$ has compact support and the support size is M_1 for a vertex v_1 of K_1^{j+1}. The scheme has *local definition* if the coefficients of the function $S^j[K,v]$ at any vertex depend only on the structure of a finite part of the complex around this vertex.

More precisely, suppose two complexes K_1^{j+1} and K_2^{j+1} for some $L_1 > M_1$ have isomorphic L_1-neighborhoods $N_{L_1}^{j+1}(v_1)$ and $N_{L_1}^{j+1}(v_2)$ of vertices $v_1 \in V_1^{j+1}$ and $v_2 = \rho(v_1) \in V_2^{j+1}$, and ρ is the isomorphism. This isomorphism maps stencil $\mathrm{St}(K_1^j, v_1)$ one-to-one to a subset of $V^{j+1}\left(N_{L_1}^{j+1}(v_2)\right)$, because the stencil is contained in $N_{M_1}^{j+1}(v_1)$. If

$$\rho(\mathrm{St}(K_1^j, v_1)) = \mathrm{St}(K_2^j, v_2)$$

and

$$S^j[K_1, v_1] \circ \rho = S^j[K_2, v_2]$$

the scheme has local definition at v_1. The last equation is shown in the following commutative diagram:

$$\begin{tikzcd}
\mathrm{St}(K_1^j, v_1) \arrow[r, "\rho"] \arrow[d, "{S^j[K_1,v_1]}"'] & \mathrm{St}(K_2^j, v_2) \arrow[dl, "{S^j[K_2,v_2]}"] \\
B &
\end{tikzcd}$$

If the set of L_1 for all possible v_1 is bounded from above by a constant L the scheme is said to be locally defined and L is called the *localization size* of the scheme.

Note the difference between this property and the compact support: For a compactly supported scheme *the value* of S^j for a given vertex depends on a finite number of values on the neighborhood. Local definition requires that the *coefficients* of the subdivision function S^j depend only on the structure part of the complex in a neighborhood of a given vertex. In our formulation local definition implies local support. This dependence could be avoided in a more complicated definition, but the difference is significant only for schemes with a small support and does not affect our constructions in any way.

In most known cases M coincides with L, but potentially they may be different. The difference is quite significant in the case of non-stationary schemes: schemes with compact support but not locally defined (for example, depending on the topology of the top level) may be quite useful.

Affine invariance with respect to the values. A natural geometric property of subdivision is invariance with respect to rotations and translations of the set of initial values. Due to linearity of subdivision, this is equivalent to requiring invariance with respect to the class of all affine transformations.

Let A be an affine transformation on B. Then a subdivision scheme S is called *affine invariant with respect to the values* if for any complex K, for any j, for any $p \in \mathcal{P}(K, B)$, $S^j[K](Ap) = AS^j[K](p)$. Above, the transformation A is applied to the values of p at the vertices of K and to the values of $S^j[K](p)$ at the vertices of $D(K)$.

The properties of subdivision schemes defined above are all independent, except the dependence between compact support and locality: a scheme can be compactly supported, and not locally defined but not the other way around. A scheme may be non-stationary, but compactly supported, etc.

Example. One of the simplest and useful subdivision schemes satisfying Conditions 1–4 is the Loop scheme (Loop [14]). The stencils and coefficients for a variant of the scheme are shown in Figure 2.2. Different stencils are used for computing values at four types of vertices:

- "old" internal: a vertex inherited from the previous subdivision level which has a closed ring of triangles around it;
- "new" internal: a vertex generated by complex refinement);
- "old" boundary: a vertex from the previous subdivision level which has an open strip of triangles around it;
- "new" boundary: a vertex inserted by complex refinement on the edge connecting two boundary "old" vertices.

This list exausts all possible positions of vertices. For example, the formula used to compute a new value at a "new" internal vertex is

$$p^{j+1}(v) = 3/8p(v_1) + 3/8p(v_2) + 1/8p(v_3) + 1/8p(v_4)$$

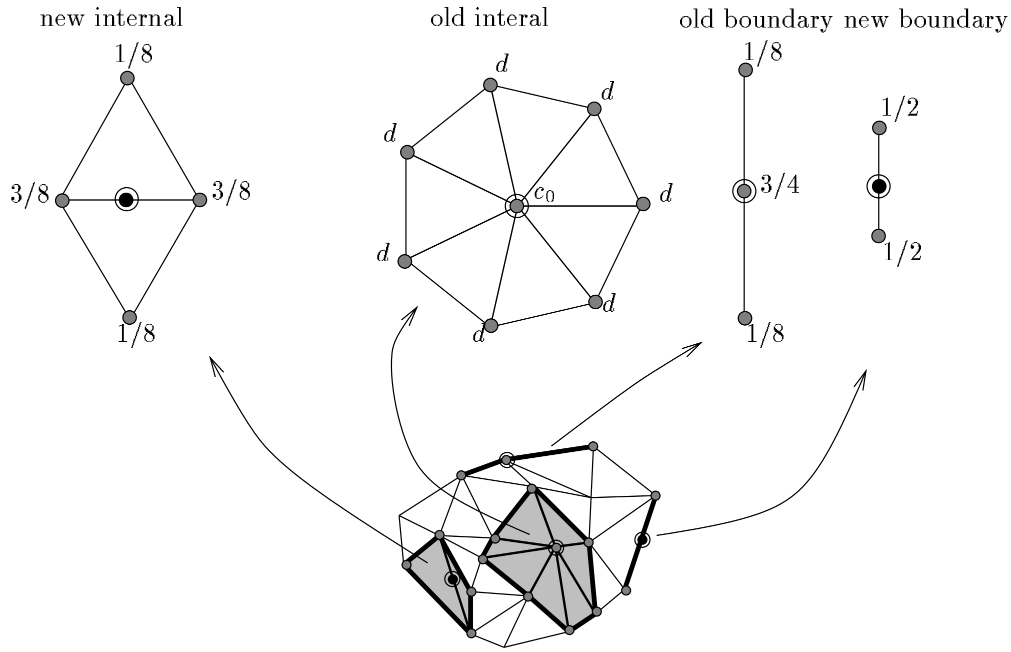

Figure 9: Loop scheme. Gray vertices of K^j are elements of $\mathrm{St}(K^j, v)$ for vertices v marked with circles. Black vertices are vertices inserted by refinement (only two are shown). The numbers next to vertices are coefficients used to compute the value $p^{j+1}(v)$ from the values $p^j(v_i)$ for $v_i \in \mathrm{St}(K^j, v)$; $c_0 = \alpha(n)/(n+\alpha(n))$, $\alpha(n) = n((5/8 - (3+2\cos(2\pi/n))^2/64)^{-1} - 1)$, $d = 1/(1+\alpha(n))$.

Vertices v_1, v_2, v_3, v_4 are the vertices of triangles adjacent to the edge of v (Figure 2.2. Details about the scheme can be found in Loop [14]).

Stationary schemes on *self-similar complexes* are of special importance for us; if a complex is self-similar, complexes K^j are isomorphic to K^0 for all j. This is true for the k-regular complexes. If a scheme is stationary, all mappings S^j can be identified for a self similar complex using the formulas in Condition 1. The scheme for such complexes is completely characterized by one mapping.

2.3 Control and Localization Sets

In this section we introduce the *control sets* of sets of vertices of subdivided complexes.

It is sufficient to define the values at the control set of a set W to compute with subdivision all values on W.

This idea generalizes the sets of control points of splines: if we know the positions of control points, all points of a spline patch can be computed.

Definition 2.7. *Define the control set of W $\mathrm{Ctrl}^j(W)$ where $W \subset V^{j+1}$ to be $V^j\left(N_M^{j+1}(W)\right)$, where M is the support size defined in Condition 2, Section 2.2.*

The values $p^{j+1}(v)$ for $v \in W$ depend only on the values $p^j(v)$ for $v \in \mathrm{Ctrl}^j(W)$.

We define $\mathrm{Ctrl}^i(W)$ for $W \in V^{j+1}$, $i \le j$ recursively:

$$\mathrm{Ctrl}^i(W) = \mathrm{Ctrl}^i(\mathrm{Ctrl}^{i+1}(W))$$

For a subset W of V^∞, the control set on level i is the union of control sets $\cup_j \mathrm{Ctrl}^i(W \cap V^j)$. For $i > j$ a control set $\mathrm{Ctrl}^i(W)$ is defined using inclusion $V^j \subset V^{i+1}$.

For subsets of $W \subset V^\infty$ we define control sets as

$$\mathrm{Ctrl}^i(W) = \cup_j \mathrm{Ctrl}^i(W \cap V^j)$$

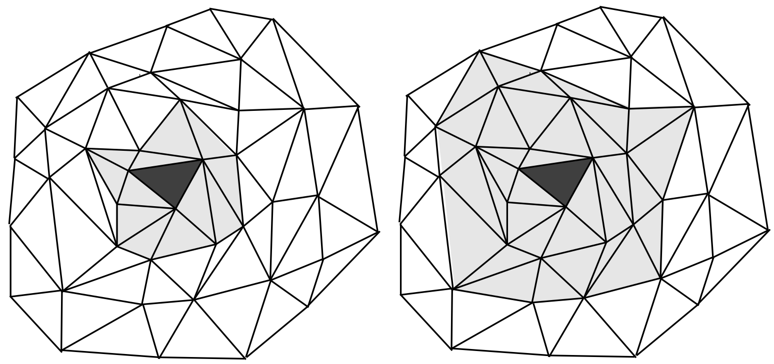

Figure 10: Left: Control set $Ctrl^0(V_T^\infty)$ for the Loop scheme, $M = 2$. Right: Control set for the Butterfly scheme, $M = 3$

Note that the control set may include vertices wich are not actually used to compute values on W; an alternative definition could be based on stencils. Our definition has the advantage of giving control sets more regular structure at the expense of increasing the size.

There is no guarantee that this set is finite. We are primarily interested in control sets of V_T^∞; $\mathrm{Ctrl}^0(V_T^\infty)$ is the direct analog of the set of control points of a spline patch; in fact, if subdivision produces a triangular spline patch (for example, Loop scheme on the regular complex), the set of values of $p^0(v)$ on $\mathrm{Ctrl}^0(V_T^\infty)$ will be exactly the set of control points of the patch.

The control sets for a triangle for the Loop scheme and for the Butterfly scheme of Dyn, Gregory and Levin [10] are shown in Figure 2.3.

Lemma 2.1. *Control set* $\mathrm{Ctrl}^j(V_T^\infty)$ *is finite for schemes with compact support and is a subset of* $V^j\left(N^j_{M-1}(V^j_T)\right)$, *where* M *is the support size of the scheme.*

Proof. We will use Lemma 2.4 from the next section.

Observe that if there is a sequence of sets C^j, $j = 0, 1\ldots$ such that for any $j > 0$ $\mathrm{Ctrl}^j(V_T^{j+1}) \subset C^j$ and $\mathrm{Ctrl}^j(C^{j+1}) = C^j$, then $\mathrm{Ctrl}^i(V_T^\infty) \subset C^i$. Indeed, $\mathrm{Ctrl}^i(V_T^{j+1}) \subset \mathrm{Ctrl}^i(C^j) = C^i$ (the last inclusion is easy to show by induction). As this is true for any j, $\mathrm{Ctrl}^i(V_T^\infty) \subset C^i$.

Take $C^j = N^j_{M-1}(V^j_T)$. Then from the definition of the control set and Lemma 2.4 we get

$$\begin{aligned}
\mathrm{Ctrl}^j(C^{j+1}) &= V^j\left(N^{j+1}_M\left(N^{j+1}_{M-1}\left(V^{j+1}_T\right)\right)\right) \\
&= V^j\left(N^{j+1}_{2(M-1)+1}\left(V^{j+1}_T\right)\right) = N^j_{M-1}\left(V^j_T\right) = C^j
\end{aligned}$$

.

It is easy to see that $M-1$ is the minimal possible size of C^j.

□

Note that both Conditions 3 and Condition 2 define a fixed-size neighborhood for any vertex in V^∞.

Exactly the same construction can be used to define *localization sets*:

Definition 2.8. *Let* L *be the localization size defined in Condition 3, Section 2.2 Define the set* $\mathrm{Loc}^j(W)$ *where* $W \subset V^{j+1}$ *to be* $V^j\left(N^{j+1}_L(W)\right)$.

Significance of localization sets is different; they can be used to establish equivalence of subdivision on parts of two complexes (Lemma 2.2).

The values $p^{j+1}(v)$ for $v \in W$ depend only on the values $p^j(v)$ for $v \in \mathrm{Loc}^j(W)$

We define $\mathrm{Loc}^i(W)$ for $W \in V^{j+1}$, $i > j$ recursively: $\mathrm{Loc}^i(W) = \mathrm{Loc}^i(\mathrm{Loc}^{i+1}(W))$. The rest of the definitions for control sets are transferred to the localization sets in the same way.

Note that the sets $\mathrm{Ctrl}^i(V_T^\infty)$ and $\mathrm{Loc}^i(V_T^\infty)$ are sets of vertices of subcomplexes. We will use the same names for these complexes.

The following fact is the basis for reducing the study of compactly supported schemes with local definition on arbitrary complexes to the case of k-regular complexes. it follows immediately from the definition of localization sets

Lemma 2.2. *Suppose K_1 and K_2 have isomorphic subcomplexes $\mathrm{Loc}^0(V_{T_1}^\infty)$ and $\mathrm{Loc}^0(V_{T_2}^\infty)$, where T_1 is a triangle of K_1 and T_2 is a triangle of K_2 and $\rho(T_1) = T_2$. If the initial values p_1 and p_2 satisfy $p_2 = \rho^*(p_1)$ on $\mathrm{Ctrl}^0(V_{T_1}^\infty)$ and $\mathrm{Ctrl}^0(V_{T_2}^\infty)$, then for all vertices v in $V_T^j \subset V_T^\infty$, $p_2^i(\rho(v)) = p_1^i(v)$ for $i \geq j$.*

A proposition similar to Lemma 2.1 holds for localization sets:

Lemma 2.3. *Localization set $\mathrm{Loc}^j(V_T^\infty)$ is finite for schemes with local definition and is a subset of $N_{L-1}^j(V_T^j)$, where L is defined in Condition 3.*

As L is always no less then M, $\mathrm{Ctrl}^j(V_T^\infty) \subset \mathrm{Loc}^j(V_T^\infty)$.

2.4 Properties of Neighborhoods

We start with several elementary properties which are straightforward to prove using the definition of the neighborhood and induction.

1. $N_1(\cup_i A_i) = \cup_i N_1(A_i)$, $A_i \subset V$
2. if $A \subset B$, $N_1(A) \subset N_1(B)$, $A, B \subset V$
3. The sets V_T^j have the following property: if $v \in V_T^{j+1} \setminus V^j$ (i.e. it is a new vertex created on jth subdivision step) then vertices $v_a, v_b \in V^j$ such that (v_i, v_a), (v_i, v_b) are edges of K^{j+1}, also are in V_T^{j+1}. This property immediately follows from the construction of V_T^j.

   The following fact is easy to show:

$$N_1^{j+1}(v) \subset N_1^{j+1}(v_a) \cup N_1^{j+1}(v_b) \tag{2.3}$$

   Using property 2 and induction, we can show that

$$N_m^{j+1}(v) \subset N_m^{j+1}(v_a) \cup N_m^{j+1}(v_b) \tag{2.4}$$

   for any m.

Now we are ready to prove the following lemma

Lemma 2.4. *For any m and any $j \geq 0$*

$$\begin{aligned} V^j\left(N_{2m}^{j+1}\left(V_T^{j+1}\right)\right) &= V^j\left(N_m^j\left(V_T^j\right)\right) \quad \textit{and} \\ V^j\left(N_{2m+1}^{j+1}\left(V_T^{j+1}\right)\right) &= V^j\left(N_m^j\left(V_T^j\right)\right) \end{aligned}$$

Proof. First we show that $D(N_m^j(W^j)) = N_{2m}^{j+1}(W^{j+1})$ for any subcomplex W^j.

A straightforward check shows that $D(N_1^j(v)) = N_2^{j+1}(v)$ for any $v \in V^j$. It is also straightforward to show that $D(\cup_i A_i) = \cup_i D(A_i)$, where A_i are subcomplexes of a complex. Therefore, for any W^j,

$$D(N_1^j(W^j)) = D(\cup_{v \in W^j} N_1^j(v)) = \cup_{v \in W^j} N_2^{j+1}(v) \tag{2.5}$$

Using (2.4) with $m = 2$, we obtain

$$D(N_1^j(W^j)) = \cup_{v \in D(W^j)} N_2^{j+1}(v) = N_2^{j+1}(W^{j+1})$$

Suppose $D(N_{m-1}^j(W^j)) = N_{2(m-1)}^{j+1}(W^{j+1})$. Then taking $\tilde{W}^j = N_{m-1}^j(W^j)$ and applying (2.4), we obtain

$$\begin{aligned} D\left(N_m^j(W^j)\right) &= D\left(N_1^j(\tilde{W}^j)\right) = N_2^{j+1}(D(\tilde{W}^j)) \\ &= N_2^{j+1}\left(N_{2(m-1)}^{j+1}(W^j)\right) = N_{2m}^{j+1}(W^{j+1}) \end{aligned}$$

which proves our proposition by induction.

Now the first formula becomes obvious: $V^j(D(A^j)) = A^j$ for any subcomplex A^j by definition of D.

Now we can prove the second formula. Let $W^{j+1} = N_{2m}^{j+1}(V_T^{j+1})$, $W^j = N_m^j(V_T^j)$.

Write $N_{2m+1}^{j+1}(V_T^{j+1})$ as $N_1^{j+1}(N_{2m}^{j+1}(V_T^{j+1})) = N_1^{j+1}(W^{j+1})$. For any vertex $v \in V^j(W^{j+1})$ $N_1^{j+1}(v)$ does not contain vertices in V^j except v. For any vertex $v \in V^{j+1}(W^{j+1}) \setminus V^j(W^{j+1})$ v_a and v_b are in $V^j(W^{j+1})$. Using (2.3), we conclude that for v $N_1(v)$ doesn't contain vertices of V^j other then v_a and v_b, which are in W^j, and therefore

$$V^j\left(N_{2m+1}^{j+1}\left(V_T^{j+1}\right)\right) = V^j\left(N_m^j\left(V_T^j\right)\right)$$

□

3 Convergence of Subdivision

The goal of this section is to introduce some basic constructions for analysis of convergence of subdivision. To define convergence, we need a parameter domain for subdivision which has topological structure.

We introduce topological complexes as parameter domains for subdivision. A topological complex for a given abstract complex is constructed as a set of planar triangles glued together along edges, one triangle per each abstract triangle of the abstract complex. if the topological complex is embedded in a Euclidean space, the topology that we describe would coincide with the induced topology.

Once topology is defined, we can talk about continuous functions on the complex. A *subdivision surface* is such a function with values in $\mathbf{R}^3$.

A simple, but very useful fact (Theorem 3.1) provides us with a representation of limit functions of subdivision as linear combinations of a set of *basis functions.* This is a straightforward generalization of a similar fact proved for regular complexes in [5].

We show that for locally defined subdivision the question of convergence and smoothness can be answered for any complex if the answer is known for k-regular complexes. This is a consequence of local definition and compact support.

3.1 Definition of Convergence

Most of the constructions in this section are routine formalizations of the intuitive concepts described in the introduction to this section.

Topological simplicial complexes. A *topological simplicial complex* $\mathcal{T}(K)$, corresponding to a given abstract simplicial complex K is defined as follows.

Let $T_{\rm std}$ be the triangle $((0,0),(1,0),(1,,1))$ in the plane. The set of triangles $F_T(K)$ consists of disjoint copies of the triangle $T_{\rm std}$, one triangle per each abstract triangle of K. $F_T(T)$ is the triangle of $F_T(K)$ corresponding to T. Each vertex v of T corresponds to a vertex $F_T(v,T)$ of $F_T(T)$. Note that each vertex v of K typically has several points of $F_T(K)$ corresponding to it: one point per each triangle of K that contains v.

An *identification equivalence relation* $\sim$ on the points of $F_T(K)$, is a set of subsets of $F_T(K)$ which are called *equivalence classes.* The points of the *identification space* $F_T(T)$ are exactly the elements of $\sim$. This is a formal equivalent of "gluing triangles together". On an identification space there is a natural topology induced by the topology of $F_T(T)$. More information on identification spaces can be found in any elementary topology textbook, such as Armstrong [1].

Any edge point y of a triangle $F_K(T)$ can be written as a linear combination of two vertices of $F_K(T)$: $y = \alpha F_K(u) + (1-\alpha)F_K(v)$. Thus, each edge point is uniquely defined by the triple (u, v, α).

We define the identification relation $\sim$ on $F_T(K)$ as follows:

- all internal points of triangles of $F_T(K)$ are equivalent only to themselves;
- Two edge points y_1 and y_2 are equivalent if and only if $u_1 = u_2$, $v_1 = v_2$, $\alpha_1 = \alpha_2$ or $u_1 = v_2$, $v_1 = u_2$, $\alpha_1 = 1 - \alpha_2$.

The topology on $\mathcal{T}(K)$ is the topology of the identification space $F_T(K)/\sim$. Let I be the identification mapping of $F_T(K)$ to $F_T(K)/\sim$, taking each point to the corresponding equivalence class. We use the following notation: for a vertex v, $\mathcal{T}(v) = I(F_T(v))$, for a triangle T, $\mathcal{T}(T) = I(F_T(T))$.

For any abstract complex neighborhood $N_m(v)$, there is a corresponding topological closed neighborhood $U_m(v) = \mathcal{T}(N_m(v))$, which is the union $\cup_i \mathcal{T}(T_i)$ for all abstract triangles in $N_m(v)$.

Refinement of topological complexes: midpoint subdivision. We mark all mappings and spaces related to K with superscript K, and those that are related to $D(K)$ are marked with D.

Refinement of abstract complexes produces a new complex $D(K)$ from an old complex K. The vertices of the K are also vertices of the new complex $D(K)$. The construction above provides us with a separate topological complex for each of K and $D(K)$, which is not very useful: our goal is to define a single parameterization domain such that there is a one-to-one correspondence between the set V^∞ and an everywhere dense subset of the domain. $\mathcal{T}(D(K))$ can be thought of as a topological complex obtained from $\mathcal{T}(K)$ by midpoint subdivision: each triangle is subdivided into four, inserting new vertices at $(1, 1/2)$, $(1/2, 0)$, $(1/2, 1/2)$. This means that all complexes $\mathcal{T}(K^j)$ can be identified, and the vertices of V^∞ on each triangle correspond to the dyadic points of that triangle.

Here is the formal construction of the homeomorphism ι^1 identifying $\mathcal{T}(K)$ with $\mathcal{T}(D(K))$.

Suppose vertices v of K have images $\mathcal{T}^K(v)$ and $\mathcal{T}^D(v)$ in $\mathcal{T}(K)$ and $\mathcal{T}(D(K))$. Let T_i, $i = 0, 1, 2, 3$ be the triangles of $D(K)$ resulting from subdivision of a triangle $T = (v_1, v_2, v_3)$ of K. Each triangle is indexed with the index of vertex of T which it contains. The triangle with index 0 is the triangle$(m_{v_1v_2}, m_{v_2v_3}, m_{v_2v_3})$. Let μ_i be the affine mapping of $F_T^D(T_i)$ into $F_T^K(T)$, such that

$$\begin{aligned}\mu_i(F_T^D(v_j)) =& \mu_i(F_T^K(v_j)) \text{ for } j = 1, 2, 3,\\ \mu_i(F_T^D(m_{v_j,v_l})) =& \frac{1}{2}\left(\mu_i\left(F_T^K(v_j)\right) + \mu_i\left(F_T^K(v_l)\right)\right), \text{ for } l \neq j\end{aligned}$$

Define ι_F from $F_T(D(K))$ to $F_T(K)$ using μ_i on each triangle T_i corresponding to a triangle T in K. As all triangles of $D(K)$ are obtained by subdivision of triangles of K, ι_F is defined everywhere on $F_T(K)$. It is easy to check that $I^K(\iota_F(t_1)) = I^K(\iota_F(t_2))$ for some t_1 and t_2 in $F_T(D(K))$ if and only if t_1 and t_2 are in the same equivalence class.

$$\begin{tikzcd} F_T(D(K)) \arrow[r, "\iota_F"] \arrow[d, "I^D"] & F_T(K) \arrow[d, "I^K"] \\ D(K) \arrow[r, "\iota^1"] & K \end{tikzcd}$$

Therefore, $I^K \circ \iota_F$ induces a one-to-one map ι^1 between $\mathcal{T}(K)$ and $\mathcal{T}(D(K))$ which is a homeomorphism. Similar homeomorphisms exist for the pairs $\mathcal{T}(K^j)$ and $\mathcal{T}(K^{j+1})$, and homeomorphisms $\iota^j : K^j \to K$ can be defined as compositions.

On each triangle of $F_T(K)$, the points corresponding to the vertices of V_T^j can be obtained by simple midpoint subdivision, i.e. for each T the points $V^\infty(T)$ are in one-to-one correspondence with dyadic points of $F_T(K)$. Clearly, if one point of an equivalence class of $\sim$ is dyadic, all of them are dyadic. Therefore, we can define the set of dyadic points of $\mathcal{T}(K)$ as the image of the set of dyadic points of all triangles in $F_T(K)$.

Thus, the vertices obtained in the process of subdivision can be identified with dyadic points of $\mathcal{T}(K)$.

From now on, we will identify vertices v of V^j with their images $\iota^j(F_T(v))$ in $\mathcal{T}(K)$. By construction of ι^j, if $v \in V^i$ and $v \in V^j$, $\iota^i(F_T(v)) = \iota^j(F_T(v))$.

Finally, we define convergence of subdivision.

For each vertex v there is a sequence of values $\{p^j(v), p^{j+1}(v), \ldots\}$ where j is the minimal number such that V^j contains v.

Definition 3.1. *A subdivision scheme is called convergent, if for any pair (K, p), $p \in \mathcal{P}(K, B)$ there is a continuous function f defined on $\mathcal{T}(K)$ with values in B, such that*

$$\lim_{i\to\infty} \sup_{v \in V^i} \left\|p^i(v) - f(v)\right\|_2 \to 0$$

The function f is called the limit function of subdivision.

Notation: $f[p]$ is the limit function generated by subdivision from the initial values $p \in \mathcal{P}(K)$.

Isomorphism of topological complexes.

Definition 3.2. *Two topological complexes $\mathcal{T}(K^1)$ and $\mathcal{T}(K^2)$ are called isomorphic if complexes K_1 and K_2 are isomorphic.*

An isomorphism ρ of abstract complexes induces a unique homeomorphism ρ' of topological complexes: as induced isomorphisms are defined for all subdivided complexes K^j, ρ' is already defined on a dense set in $\mathcal{T}(K)$. It is easy to show that ρ coincides with the restriction to V^∞ of a mapping of topological complexes which is an affine mapping for each triangle. Uniqueness of continuous extension of ρ to the topological complex immediately follows from density of V^∞ in $\mathcal{T}(K)$. We will use the same letter for the extension of an isomorphism of abstract complexes to the topological complex.

Identification of k-regular topological complexes with the plane. Any k-regular topological complex can be identified with the plane.

Suppose κ_{kF} maps each triangle of $F_T(\mathcal{R}_k)$ to the plane using the affine mapping taking $v_{0,0,0}$ to the point $(0,0)$ and $v_{i,j,l}$ to the point $R_i[j-l+l\cos 2\pi/k, l\sin 2\pi/k]^T$, where R_i is the rotation by $2i\pi/k$ around zero. It is easy to see that κ_{kF} is consistent with the equivalence relation $\sim$ and the factor-mapping κ_k is a homeomorphism between $\mathcal{T}(\mathcal{R}_k)$ and the plane.

This identification is shown in Figure 2.1.

Homeomorphism κ_k is a linear transformation on each triangle of the complex. From now on, we will assume this identification and whenever a function $f : \mathcal{R}_k \to B$ is defined on a topological k-regular complex or its subset, it is identified with the function $f \circ \kappa_k^{-1} : \mathbf{R}^2 \to B$ on the plane.

The vertices of subdivided complexes $\mathcal{T}(\mathcal{R}_k^j)$ are identified with points of the plane obtained by midpoint subdivision. We will call these points of the plane *k-dyadic.*

Note that the isomorphism $\rho_{\mathcal{R}_k}$ defined in Section 2.2, becomes the restriction to the set of vertices of $\mathcal{R}_k^1$ of the scaling of the plane $\sigma : \mathbf{R}^2 \to \mathbf{R}^2, \sigma(y) = 2y$. The corresponding homeomorphism of topological complexes is exactly σ.

3.2 Basis Function Decomposition

In this section we will derive a decomposition of the limit function of converging subdivision into the sum of *basis functions,* which can be obtained as limit functions of subdivision with initial value 1 at one of the vertices of a complex and 0 at all other vertices. The lemmas in this section are proved with the only assumption that the stencil of the subdivision scheme is finite for any point of any complex. The scheme is not required to be either compactly supported or locally defined.

First, we show that linear and piecewise constant approximations to the limit function defined below converge to the limit function in L_∞ norm on any compact subset of $\mathcal{T}(K)$.

Definition 3.3. *Let $T = (v_1, v_2, v_3)$ be a triangle of K^j, $y \in \mathcal{T}(T)$. Let $\alpha_1, \alpha_2, \alpha_3$ be the barycentric coordinates of y in the triangle T, $\alpha_1+\alpha_2+\alpha_3 = 1$. Define $L[p^j](y) = \alpha_1 p^j(v_1)+\alpha_2 p^j(v_2)+\alpha_3 p^j(v_3)$, jth linear approximation of $f[p^0](y)$.*

Clearly, $L[p^j](y)$ is continuous, and the definition is consistent on common edges of triangles.

Lemma 3.1. *On any compact subset A of K*

$$\lim_{j\to\infty} ||L^j(y) - f[p^0](y)||_{\infty,A} = 0 \tag{3.1}$$

Proof. We will denote $f[p^0](y)$ by $f(y)$, $L[p^j](y)$ by $L^j(y)$. For any compact set $A \subset \mathcal{T}(K)$ there is only a finite number of triangles T of K^j such that $A \cap \mathcal{T}(T) \neq \emptyset$. Thus,

$$||L^j(y) - f(y)||_{\infty,A} < \max_{\mathcal{T}(T)\cap A \neq \emptyset} ||L^j(y) - f(y)||_{\infty,T}$$

and it is sufficient to show convergence only for $A = \mathcal{T}(T)$, where

T is a triangle of K, $T = (v_1, v_2, v_3)$.

$$\begin{aligned}
\left\| L^j(y) - f(y) \right\|_\infty &\leq \max_{s \in \mathcal{T}(T)} \max_{y \in \mathcal{T}(T)} \left| L^j(y) - L^j(s) \right| \\
&\leq \max_{s \in \mathcal{T}(T)} \max_{i \in \{1,2,3\}} \left| L^j(v_i) - f(s) \right| \\
&= \max_{s \in \mathcal{T}(T)} \max_{i \in \{1,2,3\}} \left| p^j(v_i) - f(s) \right|
\end{aligned}$$

As $\mathcal{T}(T)$ is compact, $f(y)$ is uniformly continuous on $\mathcal{T}(T)$ in the standard metrics on T, i.e. for any $\epsilon > 0$ there is a $\delta > 0$ such that for any s_1, s_2 satisfying $|s_1 - s_2| < \delta$, $|f(s_1) - f(s_2)| < \epsilon$.

As the maximal distance between points of $\mathcal{T}(T)$ for a triangle T of K^m is $\frac{\sqrt{2}}{2^m}$, then for any $\epsilon > 0$ there is m_1 such that for all triangles T of K^j, $j > m_1$, for any $s_1, s_2 \in \mathcal{T}(T)$, $\left|f(s_1) - f(s_2)\right| < \epsilon$.

By definition, $f(v_i) = \lim_{l \to \infty} p^l(v_i)$. Therefore, there is a number m_2 such that for all $l > m_2$ $\left|p^l(v_i) - f(v_i)\right| < \epsilon$ for $i \in \{1, 2, 3\}$.

Let $m = \max(m_1, m_2)$. Then for $j > m$

$$\max_i \left|p^j(v_i) - f(s)\right| < \max_i (\left|p^j(v_i) - f(v_i)\right| + |f(v_i) - f(s)|) < 2\epsilon$$

We conclude that (3.1) is true.

□

A similar statement holds for piecewise-constant approximations, defined in the following way.

Definition 3.4. *Suppose for each triangle T of K^j we choose a vertex v, and define $C^j(t)$, for t in the interior of $\mathcal{T}(T)$ to be equal to $p^j(v)$. On the intersections of triangles $C^j(y)$ is defined by arbitrarily picking a triangle "responsible" for the intersection.*

Lemma 3.2. *On any compact subset A of K*

$$\lim_{j \to \infty} ||C^j(y) - f[p^0](y)||_{\infty, A} = 0 \tag{3.2}$$

The proof of this lemma is exactly the same as the proof of Lemma 3.1.

Now we can prove the following theorem:

Theorem 3.1. *Suppose a subdivision scheme S converges on a complex K for any initial values $p^0 \in \mathcal{P}(K, \mathbf{R})$ Then*

$$f[p^0](y) = \sum_{v \in V^0} p^0(v)\phi_v(y) \tag{3.3}$$

where $\phi_v(y)$ is continuous and $\phi_v(y) = f[\delta_v](y)$, $\delta_v(v) = 1$ and $\delta_v(w) = 0$ for $w \neq v$.

Proof. It follows from Lemma 3.2 that

$$\phi_v(y) = \lim_{j \to \infty} L^j[\delta_v](y) \tag{3.4}$$

For any vertex $v \in V^j$,

$p^j(v) = \sum_{w \in V^{j-1}} a^{j-1}_{wv} p^{j-1}(w)$, where only a finite number of $a^{j-1}_{wv} \neq 0$.

By induction, $p^j(v) = \sum_{w \in V^{j-l}} a^{j-l}_{wv} p^{j-l}(w)$ for all $l < j$, i.e. $p^j(v) = \sum_{w \in V^0} a^0_{wv} p^0(w)$, with only finite number of $a^0_{wv} \neq 0$.

In particular, for $p^0 = \delta^0_u$, $\delta^j_u(v) = a_{uv}$, i.e.

$$p^j(v) = \sum_{w \in V^0} \delta^j_w(v) p^0(w)$$

$L[p^j](y) = \alpha_1 p^j(v_1) + \alpha_2 p^j(v_2) + \alpha_3 p^j(v_3)$, if $y \in \mathcal{T}(T)$, T is a triangle of K^j and $\alpha_1, \alpha_2, \alpha_3$ are barycentric coordinates of y. Then

$$L^j[p](y) = \sum_{w \in V^0} \left(\alpha_1 \delta_w^j(v_1) + \alpha_2 \delta_w^j(v_2) + \alpha_3 \delta_w^j(v_3) \right) p^0(w) = \sum_{w \in V^0} p^0(w) L[\delta_w^j](y)$$

The limits of the right and of the left side exist by Lemma 3.1. Taking limits and using (3.4), we get (3.3). □

3.3 Local properties of surfaces.

Intuitively it is clear that smoothness of a surface is a local property, that is, if a surface is smooth at an arbitrarily small neighborhood of a point, it is smooth at that point.

To make this idea precise, we need to define what we mean by a point of a surface and by smoothness in a neighborhood of the point.

Definition 3.5. *Let X be a topological 2-manifold, possibly with boundary, and f be a mapping from X to $\mathbf{R}^3$. A point of the surface $f(X)$ is a pair $(x, f(x))$, where $x \in X$.*

A neighborhood of a point $(x, f(x))$ of a surface is a neighborhood $U(x)$ in X together with its image $f(U)$.

It would be tempting to get rid of the parameterization given by subdivision and consider only the image $f(X)$. The problem with this approach is that it is too restrictive: for example, it doesn't seem to be possible to define smoothness on the subsets of $\mathbf{R}^3$ in such a way that an intuitively smooth surface with self-intersections is formally smooth (Figure 3.3). If we want a definition that works in such situations, we need to consider the points of the self-intersection to be sets of points, one point per each "sheet" that goes through the point. A natural solution for this problem is to consider parametric surfaces, defined as above.

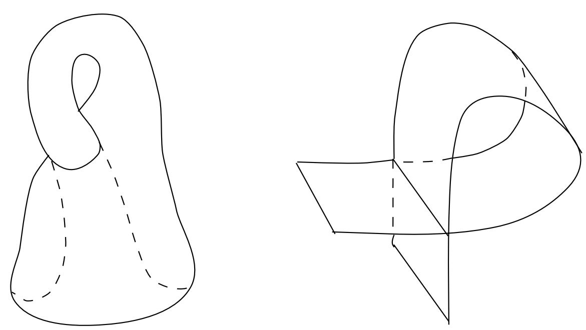

Figure 11: Self-intersecting surfaces.

Now we can define a C^k-continuous surface.

Definition 3.6. *A surface is C^k-continuous, at a point $(x, f(x))$ if there is a neighborhood $U(x)$ of the point $x \in X$, such that the subset of $\mathbf{R}^3$ $f(U)$ admits a C^k-continuous regular parameterization.*

Note that we only need a continuous parameterization over a topological space to be able to define "a small piece" of surface; local C^k-parameterization which is required for smoothness can be completely independent.

3.4 Reduction to the Stationary Case

In this section we reduce local analysis of the limit surfaces generated by a scheme satisfying Conditions 2 and 3 to the analysis of limit surfaces on k-regular complexes.

Suppose we can demonstrate that for a certain subcomplex K_y such that y is an interior point of $\mathcal{T}(K_y)$, there is an isomorphism ρ_y of K_y and K_{std} where K_{std} is a subcomplex of a standard complex, and any limit function $f[p]$, $p \in \mathcal{P}(K)$ can be represented locally on $\mathcal{T}(K_y)$ as $f[\rho_y^*(p)] \circ \rho_y$:

$$\begin{tikzcd} \mathcal{T}(K_y) \arrow[r, "f[p]"] \arrow[d, "\rho_y"] & \mathcal{T}(K_{std}) \arrow[dl, "{f[\rho_y^*(p)]}"] \\ B & \end{tikzcd} \tag{3.5}$$

We will use the k-regular complexes, possibly with boundary, as K_{std}. Then analysis of all local properties of f, of C^k-continuity in particular, can be done on k-regular (possibly with boundary) complexes.

In order to demonstrate existence of the isomorphisms ρ_y described above, we consider several types of points of $\mathcal{T}(K)$, shown in Figure 3.4.

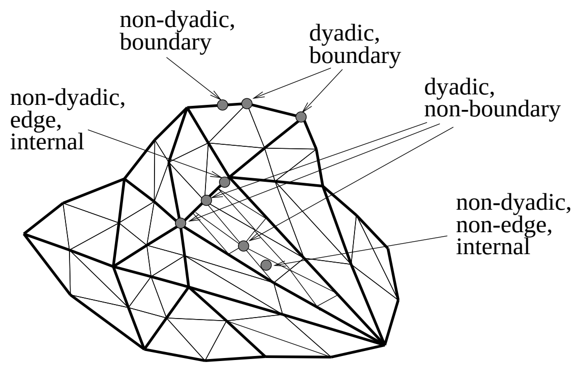

Figure 12: Types of points of $\mathcal{T}(K)$. Non-dyadic vertices are not vertices of a triangle on any level of subdivision, not only on the levels shown in the picture.

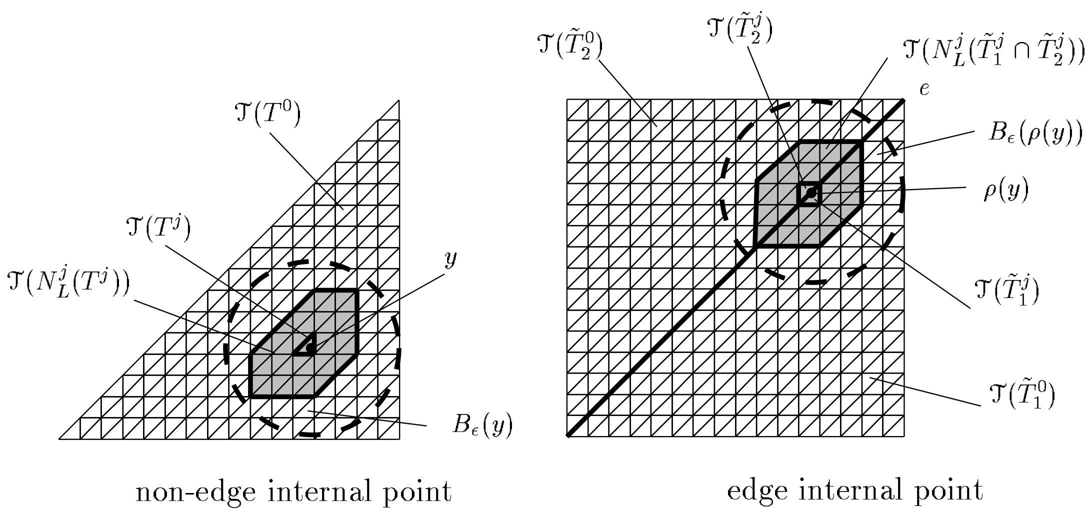

Figure 13: Non-dyadic points, reduction to the regular case. L=3

Non-dyadic, non-edge, internal. Suppose ϵ is the distance from y to the boundary of $\mathcal{T}(T^0)$. Let T^j be the triangle of K^j such that $y \in \mathcal{T}(T^j)$. There is j such that $\mathcal{T}(N_{L-1}^j(T^j)) \subset B_\epsilon(y)$, where B_ϵ is a disk of radius ϵ centered at y. Therefore, $N^j_{L-1}(T^j)$ doesn't contain any extraordinary vertices and it is isomorphic to a subcomplex of the regular complex. Explicit enumeration of the vertices in $N^j_{L-1}(T^j)$ establishes the isomorphism. The fact that the localization set doesn't contain extraordinary vertices guarantees that the mapping induced by the isomorphism satisfies (3.5).

Non-dyadic, edge, internal. Suppose $y \in \mathcal{T}(T^0_1) \cap \mathcal{T}(T^0_2)$. For any j, there is an isomorphism ρ between $D^j(T^0) \cup D^j(T^1)$ and the subcomplex of the regular complex consisting of $v_{i,k}$, $0 \le i, k \le 2^j$,induced by the map from $\mathcal{T}(T^0_1) \cup \mathcal{T}(T^0_2)$ to the square $[0,1] \times [0,1]$, which is identity for T^0_1 and reflection across $e = [(0,0),(1,1)]$ for T^0_2, assuming that e is the edge of T^0_1 identified with the same edge of T^0_2. Let $\tilde{T}^0_i = \rho(T^0_i)$, $i = 1, 2$.

Suppose ϵ is the distance from $\rho(y)$ to the boundary of the square formed by $\tilde{T}^0_1$ and $\tilde{T}^0_2$. Then there is j such that $N^j(\tilde{T}^j_1 \cup N^j(\tilde{T}^j_2) \subset B_\epsilon(\rho(y))$. Then we proceed as in case 1.

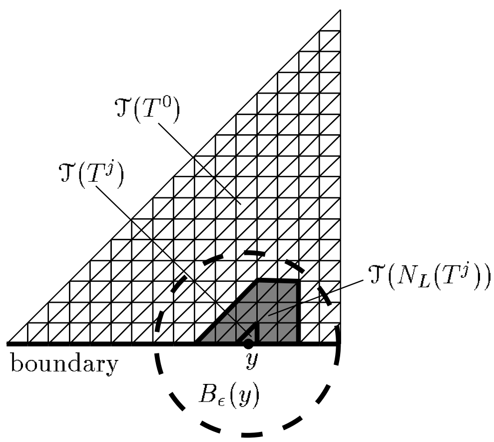

Figure 14: Non-dyadic boundary point, reduction to the stationary case. $L = 3$.

Non-dyadic, boundary. Suppose $y \in \mathcal{T}(T^0)$. Assume $[(0,0),(1,0)]$ is on the boundary. Map T^0_1 to the triangle (v_{00}, v_{01}, v_{11}) in the regular complex with boundary. The rest of the argument is similar to case 1.

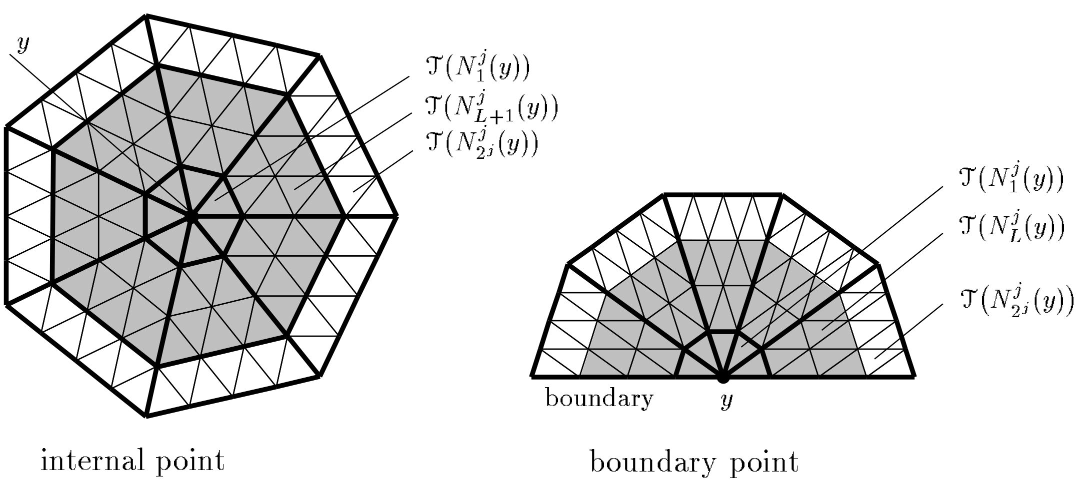

Figure 15: Dyadic points, reduction to the stationary case. L=3, j =2.

Dyadic, internal. Clearly, $N_1^0(y)$ is isomorphic to N_1^0 in the k-regular complex, where k is the valence of the vertex. Therefore, $N_{2^j}^j$ in K is isomorphic to a similar neighborhood in the k-regular complex $\mathcal{R}_k$. Choosing sufficiently large j, we obtain the required isomorphism.

Dyadic boundary. This case is similar to the previous one, but the k-regular complex with boundary is used instead of the k-regular complex.

In the case of internal non-dyadic and regular dyadic points of $\mathcal{T}(K)$, it is sufficient to analyze smoothness of the scheme on the regular complex.

In the case of internal extraordinary points, it is sufficient to analyze smoothness on the k-regular complexes.

Thus, analysis of smoothness of the scheme at internal points of $\mathcal{T}(K)$ is reduced to two cases: the regular case, for which the scheme is defined with a finite mask, and k-regular case, for which the scheme is defined with a finite number of finite masks, corresponding to different symmetry classes of vertices.

On the boundary, it is also sufficient to analyze two cases: regular and k-regular complexes with boundary.

The regular case is described in detail for functional subdivision by Cavaretta, Dahmen and Micchelli [5].

The second case did not receive much attention until recently. Relevant literature includes works by Loop, Warren and Kobbelt [14, 24, 13].

4 Subdivision Matrix

In this section we continue to prepare a foundation for C^k-continuity criteria. We introduce the subdivision matrix, which is the matrix that relates the values on the localization set of the topological 1-neighborhood of zero in the k-regular complex to the values on the similar neighborhood on the next subdivision level. This matrix is defined only for stationary schemes.

As the localization set of 1-neighborhood of zero $\dot{U}_1$ is contained in N_L, the limit function on $\dot{U}_1$ is completely defined by values p^0 on N_L. In particular, if we take the initial values to be equal to the values of an eigenvector of the subdivision matrix, subdivision will produce a limit function called *eigenbasis function.* These functions satisfy scaling relations which will be used in the next section to derive a criterion for C^k-continuity.

We introduce layers of a k-regular complex, which are rings of triangles around a vertex, where only regular subdivision rules are used.

Finally, we prove a simple condition on the eigenvalues of the subdivision matrix that is sufficient for convergence of subdivision.

4.1 Subdivision Matrix and Layers

Let L be the localization size for $\mathcal{S}$.

Definition 4.1. *Consider $N^0_{L-1}(N^0_1) = N^0_L$. in the k-regular complex $\mathcal{R}_k$. As $N^j_L \simeq N^{j+1}_L$ and $\mathrm{Ctrl}^j(N^{j+1}_L) \subset N^j_L$, the values of $p^{j+1}|_{N^{j+1}_L}$ can be computed from $p^j|_{N^j_L}$ only, using a matrix S. We will call this matrix the subdivision matrix of $\mathcal{S}$ on $\mathcal{R}_k$.*

The fact that the natural size of the subdivision matrix turns out to be exactly L is quite remarkable: there is no obvious reason for this to be true.

Example. The following matrix is the subdivision matrix for the Loop scheme on the 3-regular complex. The vertices are indexed as it is shown in Figure 4.1.

$$\left(\begin{array}{c|ccc|ccc|ccc}
7/16 & 3/16 & 3/16 & 3/16 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
3/8 & 3/8 & 1/8 & 1/8 & 0 & 0 & 0 & 0 & 0 & 0 \\
3/8 & 1/8 & 3/8 & 1/8 & 0 & 0 & 0 & 0 & 0 & 0 \\
3/8 & 1/8 & 1/8 & 3/8 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
1/16 & 5/8 & 1/16 & 1/16 & 1/16 & 0 & 0 & 1/16 & 0 & 1/16 \\
1/16 & 1/16 & 5/8 & 1/16 & 0 & 1/16 & 0 & 1/16 & 1/16 & 0 \\
1/16 & 1/16 & 1/16 & 5/8 & 0 & 0 & 1/16 & 0 & 1/16 & 1/16 \\ \hline
1/8 & 3/8 & 3/8 & 0 & 0 & 0 & 0 & 1/8 & 0 & 0 \\
1/8 & 0 & 3/8 & 3/8 & 0 & 0 & 0 & 0 & 1/8 & 0 \\
1/8 & 3/8 & 0 & 3/8 & 0 & 0 & 0 & 0 & 0 & 1/8
\end{array}\right)$$

Conditions 1–4 from Section 2 are reflected in the properties of the matrix. For example, Condition 4 (affine invariance) implies that the subdivision matrix has eigenvalue 1 with eigenvector $x(v) = 1$ for all $v \in N_L$. Indeed, let $p'(v) = p(v) - 1$ for $p \in N_m$, i.e. $p' = p - x$. Then $Sp' = Sp - x$. But by linearity $Sp' = Sp - Sx$, thus $Sx = x$.

Next we define the *layers* of k-regular complexes. For any triangle T, $T \cap N_L = \emptyset$, convergence of $\mathcal{S}$ on $\mathcal{T}(T)$ is equivalent to convergence of $\mathcal{S}$ on a triangle of the regular complex (Lemma 2.3). By assumption, $\mathcal{S}$ converges there. Thus the scheme converges on $\mathcal{R}_k \setminus N_L$. Note that $D(\mathcal{R}^j_k \setminus N^j_L) = \mathcal{R}^{j+1}_k \setminus N^{j+1}_{2L}$.

Definition 4.2. *A Layer Lr^j of the subdivided k-regular complex $\mathcal{R}^j_k$ is defined as $N^j_{2L} \setminus N^j_L$.*

The union of all layers covers a neighborhood of zero in $\mathcal{T}(\mathcal{R}_k)$, except for zero itself:

$$\bigcup_{j=0}^{\infty} \mathcal{T}(\mathrm{Lr}^j) = \dot{U}^0_{2(L+1)} \tag{4.1}$$

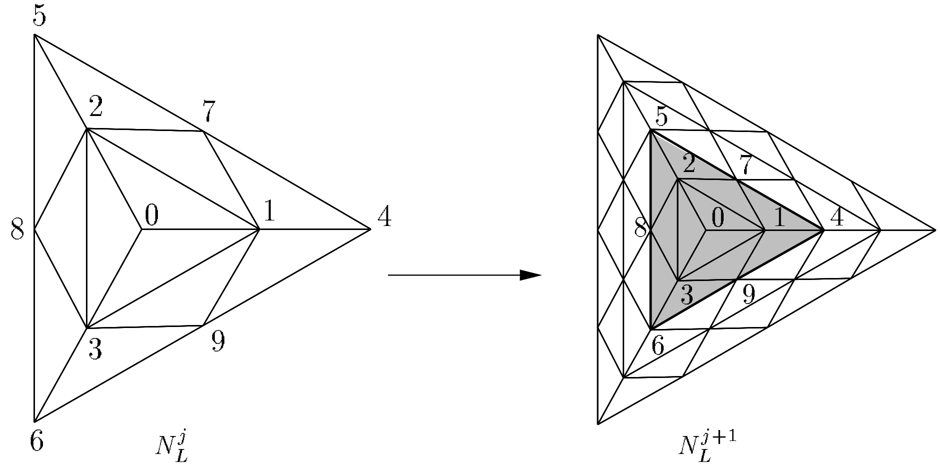

Figure 16: Subdivision matrix operates on the values of p^j on N_L^j (left) and produces the values of p^{j+1} on N_L^{j+1} (right). In the picture, $L = 2$, as for the Loop scheme. The numbers shown in the picture are used to arrange the values at vertices of N_L^j into one vector.

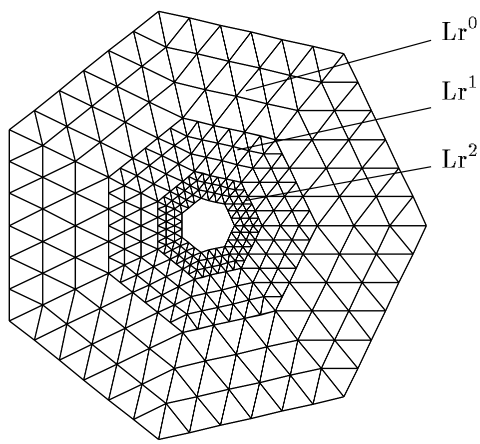

Figure 17: Layers Lr^0,Lr^1,Lr^2 for $L = 3$ (Butterfly scheme).

We will use notation $\dot{U}_l^j$ for $U_l^j \setminus \{(0,0)\}$.

By construction of the layers functions $f[p^n]$ are defined on $\cup_{j=0}^m \mathrm{Lr}^j$ for $n \geq m$, and $f[p^n] = f[p^m]$ on $\cup_{j=0}^m \mathrm{Lr}^j$. Thus, we can define $f^\circ(y) = f[p^n](y)$ for $y \in \cup_{j=0}^n \mathrm{Lr}^j$. If the subdivision scheme converges on $\mathcal{R}_k$ the limit function $f[p^0]$ coincides with $f^\circ[p^0]$ on $\mathcal{T}(N_{2L}^0)$.

We summarize the observations above in the following

Lemma 4.1. *On a k-regular complex $\mathcal{R}_k$ for a subdivision scheme $\mathcal{S}$ converging on the regular complex, for any $p \in \mathcal{P}(N_L)$ there is a continuous limit function $f^\circ[p]$ defined on $\dot{U}_1$; at dyadic points v, the sequences of values $p^i(v)$ converge to $f^\circ[p]$. If the scheme converges on $\mathcal{R}_k$, $f^\circ[p]$ coincides with $f[p]$ on $\dot{U}_1$.*

4.2 Eigenbasis Functions and Scaling Relations

The limit functions generated by the eigenvectors of the subdivision matrix have an important property: they satisfy scaling relations described in this section. This property will be used in Section 5 to derive a criterion for C^k-continuity of subdivision.

Definition 4.3. *Suppose S is the subdivision matrix of a subdivision scheme $\mathcal{S}$ on the k-regular complex, and x_{im} are the vectors of the Jordan basis of S. Then the limit functions $f[x_{im}](y)$ defined on U_1 are called* eigenbasis functions.

Eigenbasis function $f[x_{ij}]$ is also denoted f_{ij}.

It immediately follows from Theorem 3.1 that any function on U_1 can be written as a linear combination

$$f[x](y) = \sum_{im} a_{im} f[x_{im}](y) \tag{4.2}$$

if $x = \sum_{im} a_{im} x_{im}$.

We assume that the coefficients of the subdivision schemes are real. In this case, it is possible to reduce the subdivision matrix to the real Jordan form with eigenbasis vectors r_{ij}, and decompose the limit function into a linear combination of *real eigenbasis functions* g_{ij}, satisfying $g_{ij} = f_{ij}$ for real eigenvalues, $g_{ij} = \Re f_{ij}, g_{\bar{i}j} = \Im f_{\bar{i}j}$ for complex λ_i and $\lambda_{\bar{i}} = \bar{\lambda}_i$.

Trivial eigenbasis functions and eigenvalues. Some of the eigenbasis functions of a subdivision scheme may be identically equal to zero. These eigenbasis functions don't affect the limit functions of subdivision in any way. If an eigenbasis function corresponds to the eigenvalue $\lambda = 0$, it also cannot affect convergence or smoothness of the scheme. Although it need not be identically zero on U_1, it is identically zero on a smaller neighborhood U_1^1. Eigenbasis functions which are identically equal to zero or correspond to the eigenvalue zero are called *trivial.*

If all eigenbasis functions corresponding to an eigenvalue λ are trivial, λ is said to be a *trivial eigenvalue.*

Typically, we exclude trivial eigenvalues from consideration. Although they can affect numerical stability of the scheme, they cannot affect smoothness of the limit surfaces in any way.

The non-trivial eigenvectors of the subdivision matrix span a subspace of $\mathcal{P}(N_L, \mathbf{R})$ or $\mathcal{P}(N_L, \mathbf{C})$ that we call the *non-trivial subspace.* The limit function generated by any $x \in \mathcal{P}(N_L, \mathbf{R})$ is equal to the limit function generated by the projection of x onto the non-trivial subspace.

Constant eigenvectors and eigenbasis functions .

Observe that an eigenvector x can produce a limit function equal to a non-zero constant only if the vector itself is constant on N_L and corresponds to the eigenvalue 1 (Lemma 6.1.) From the same argument it follows that any vector corresponding to the eigenvalue 1 should be constant. By affine invariance, there is always an eigenvector which is constant corresponding to the eigenvalue 1. This means that the eigenvalue 1 has one-dimensional eigenspace.

Suppose the set of vectors x_m $m = 0 \dots n$ are generalized eigenvectors of S for the eigenvalue λ, $Sx_m = \lambda x_m + x_{m-1}$ for $m > 0$, $Sx_0 = \lambda x_0$.

In Section 3.1 we have introduced an identification of the k-regular complex and the plane. Recall that the map $y \to y/2$ of the plane to itself maps $N_m^j \to N_m^{j+1}$.

Theorem 4.1. *For a subdivision scheme $\mathcal{S}$ the functions $f[x_m] = f^m$ defined on $\dot{U}_1$ in the k-regular complex $\mathcal{R}_k$ satisfy*

$$f_m(y/2) = \lambda f_m(y) + f_{m-1}(y) \tag{4.3}$$

Proof. By linearity of subdivision, if $x_m^j = \lambda x_m^j + x_{m-1}^j$, $x_m^j \in \mathcal{P}(D^j(N_L^0))$, then $x_m^{j+1} = \lambda x_m^{j+1} + x_{m-1}^{j+1}$, $x_m^{j+1} \in \mathcal{P}(D^{j+1}(N_L^0))$,

As $L_m^j(y) = L[x_m^j](y)$ is linear in y on each triangle $\mathcal{T}(T)$, where T is a triangle of $D^j(N_1)$, then

$$L_m^j(\frac{y}{2}) = \lambda L_m^j(y) + L_{m-1}^j(y) \tag{4.4}$$

For any complex $\mathcal{R}_k^p \setminus N_L^p$ $\lim_{j\to\infty} L_m(y) = f(y)$. Therefore, $\lim_{j\to\infty} L_m(y) = f(y)$ everywhere, and (4.3) follows from (4.4).

□

4.3 Jordan Normal Form of the Subdivision Matrix

Let $\lambda_0, \lambda_1, \lambda_2 \ldots, \lambda_J$ be the eigenvalues of S satisfying $|\lambda_0| \geq |\lambda_1| \geq \ldots |\lambda_J|$.

Let $\{x_{im} | i = 0 \ldots J, m = 0 \ldots n_i\}$ be the basis in which the subdivision matrix S has Jordan normal form. Any vector x can be written in the form

$$x = \sum_{i=0}^{J} \sum_{m=0}^{n_i} a_{im} x_{im},$$

and x_{im} satisfies $S x_{im} = \lambda_i x_{im} + x_{i(m-1)}$ for $m \geq 0$, $S x_{i0} = \lambda_i x_{i0}$.

We will need an expression for the vector $S^l x$. We will assume that $l \geq n_i$ for all i.

First, we consider a matrix with one Jordan block. In this case the subscript i can be dropped.

We introduce a formal operator N, acting on eigenvectors of S. $N x_m = x_{m-1}$ for $m > 0$ and $N x_0 = 0$. The action of S is just $\lambda I + N$. Clearly, multiplication by a constant and N commute. Therefore,

$$S^l x = (\lambda I + N)^l \sum_{m=0}^{n} a_m x_m = \sum_{m=0}^{n} \sum_{q=0}^{j} \binom{l}{q} \lambda^{l-q} x_{j-q} a_j$$

Changing indexing and the order of summation, we get

$$\sum_{j=0}^{n} \sum_{q=0}^{j} \binom{l}{q} \lambda^{l-q} x_{j-q} a_j = \sum_{m=0}^{n} \sum_{q=0}^{m} \binom{l}{m-q} \lambda^{l-m+q} x_q a_m = \sum_{q=0}^{n} x_q \sum_{m=0}^{n-q} \binom{l}{m} \lambda^{l-m} a_{m+q}$$

Thus,

$$S^l x = \lambda^{l-n} \sum_{q=0}^{n} x_q \left(\sum_{m=0}^{n-q} \binom{l}{m} \lambda^{n-m} a_{m+q} \right) \tag{4.5}$$

In the general case,

$$S^l x = \sum_{i=0}^{J} \lambda_i^{l-n_i} \sum_{q=0}^{n_i} x_{iq} \alpha_{iq}^l \tag{4.6}$$

where $\alpha_{iq}^l = \sum_{m=0}^{n_i-q} \binom{l}{m} \lambda^{n_i-m} a_{i(m+q)}$.

Estimating $|\alpha_{iq}^l|$, we get

$$|\alpha_{iq}^l| \leq \max_m a_{im} \sum_{m=0}^{n_i-q} \binom{l}{m} \lambda^{n_i-m} \leq \max_m |a_{im}| \sum_{j=0}^{n_i-q} \binom{l}{m}$$

for $|\lambda_i| \leq 1$.

$$|\alpha_{iq}^l| \leq \max_m |a_{im}| l^{n_i-q+1} = O(l^{n_i-q+1})$$

Finally,

$$S^l x = \sum_{i=0}^{J} \lambda_i^{l-n_i} \sum_{q=0}^{n_i} x_{iq} O(l^{n_i - q+1}) \tag{4.7}$$

In certain cases we have to use bases consisting of real vectors. Generalized eigenvectors corresponding to complex eigenvalues can be complex. As the matrix is real, complex eigenvalues came in conjugate pairs and the same is true for complex eigenvectors. Further, if the decomposition (4.5) is written for a real vector x, the coefficients of a pair of conjugate vectors of the Jordan basis are complex conjugate.

Let $\lambda = |\lambda| \exp(i\phi_\lambda)$, $a_{m+q} = |a_{m+q}| \exp(i\phi_{m+q})$ and $x_q^i = |x_q^i| \exp(i\psi_q^i)$.

Using Equation (4.6) for a matrix with 2 Jordan blocks corresponding to λ and $\boldsymbol{\lambda}$, we obtain an analog of (4.5).

$$[S^l x]^i = 2|\lambda^{l-n}| \sum_{q=0}^{n} |x_q^i| \sum_{m=0}^{n-q} \binom{l}{m} |\lambda^{n-m}| |a_{m+q}| \cos(\phi_\lambda (l-m) + \phi_{m+q} + \psi_q^i) \tag{4.8}$$

This decomposition of x is real.

4.4 Convergence Criterion for Subdivision

The idea of layers introduced in Section 4.1 can be used to prove a criterion for convergence of subdivision. This criterion is very similar to Theorem 3.2 [19] by Reif. We don't make any assumptions about the range of p and f — it can be $\mathbf{R}^n$ or $\mathbf{C}^n$. For the second part of the lemma, we use a fact that will be proven in Section 6.

Suppose $\lambda_1 \dots \lambda_n$ are non-trivial eigenvalues of the subdivision matrix and $|\lambda_1| \geq \dots \geq |\lambda_n|$.

Theorem 4.2. *Suppose $\mathcal{S}$ converges on the regular complex. $\mathcal{S}$ converges on $\mathcal{R}_k$ if and only if the non-trivial eigenvalues of the subdivision matrix S are all less then one, except $\lambda_0 = 1$.*

Proof. Sufficiency. Suppose $\lambda_0 = 1$, and all non-trivial eigenvalues satisfy $|\lambda_i| < 1$. As it was shown in Section 4.2, λ_0 has only one eigenvector x_0, which is constant on N_L.

Consider a sequence of points $s_i \in \mathcal{R}_k$ such that $\lim_{i\to\infty} s_i = 0$. By Lemma 4.1, the limit function $f[p](y)$ is defined everywhere except 0. For a triangle T of $\mathcal{R}_k$ such that $\mathrm{Loc}^j(\mathcal{T}(T))$ doesn't contain 0,

$$f[p](y)\,|_{\mathcal{T}(T)} = \sum_{v \in \mathrm{Ctrl}^j(\mathcal{T}(T))} p(v)\phi_v(y)$$

Suppose that for all v $\|p(v) - a\|_2 < \epsilon$ for some a. For any y, $\|\sum_v p(v)\phi_v(y) - a\|_2 = \|\sum_v (p(v) - a)\phi_v(y)\|_2$ by affine invariance.

Hence,

$$\left\| \sum_v p(v)\phi_v(y) - a \right\|_2 \leq \epsilon \sum_v |\phi_v(y)| < \epsilon C \tag{4.9}$$

where $C = \max_{y \in \mathcal{T}(T)} \sum_v |\phi_v(y)|$. C is finite, because $\phi_v(y)$ are continuous and $\mathcal{T}(T)$ is compact. Clearly, (4.9) is also true for any union of triangles T.

Consider the sequence p^j, $p^j \in \mathcal{P}(N^j_{2(L+1)+M})$. The control set $\mathrm{Ctrl}(\mathcal{T}(Lr^j))$ is contained in $\mathcal{P}(N^j_{2(L+1)+M})$: the limit function restricted to $\mathcal{T}(\mathrm{Lr}^j)$ depends only on the values of p^j.

There is a matrix S_{lr} such that $S_{lr}p^j = p^{j+1}$. As the subdivision matrix takes $\tilde{p}^j = p^j|_{N_L}$ to $\tilde{p}^{j+1}$, eigenvalues of S_{lr} are also eigenvalues of S and have to satisfy the conditions of the theorem.

Let x_{im}, $i \in \{0 \dots J\}$, $m \in \{0 \dots n_i\}$, be the the Jordan basis of S_{lr}. Write

$$p^j = a_0 x_0 + \sum_{i=1}^{J} \lambda_i^{l-n_i} \sum_{q=0}^{n_i} x_{iq} O(l^{n_i - q + 1}),$$

using(4.7) and $\lambda_0 = 1 > |\lambda_1|$.

Thus, as $j \to \infty$ $p^j \to a_0 x_0$; in particular, $\lim_{j\to\infty} p^j(0) = f[p_0](0) = a_0$

For any $\epsilon > 0$ there is l such that $\|s_j\|_2 < \epsilon$ for $j > l$. Therefore, for any n there is l such that $s_j \in \cup_{i=n}^{\infty} \mathcal{T}(\mathrm{Lr}^i)$ for for $j > l$, unless $s_j = 0$.

Choose l' so that $\left\|p^j - a_0 x_0\right\|_\infty < \delta/C$, for a given δ. Then $\|f[p](s) - a_0\|_2 < \delta$ for $s \in \mathrm{Lr}^{n'}$ for $n' > l'$; if $s_j = 0$, this is also true.

We conclude that that $f[p](s_j)$ converges to a_0, and the scheme converges on $\mathcal{R}_k$.

Necessity. Eigenvalue 1 should be a non-trivial eigenvalue of the matrix by affine invariance. Suppose $|\lambda_0| > 1$ or $|\lambda_1| = 1$, but $\lambda_1 \neq 1$. Consider the function $f[x_{1j}](t)$ generated by corresponding eigenvector, which is defined on $\dot{U}_1$. It is not continuous unless it is identically 0 by Lemma 6.1, therefore, λ_1 is trivial.

$\square$

5 C^k-continuity of Subdivision

In this section we formulate some of our main results on C^k-continuity of subdivision. These results are based on the properties of the functions satisfying scaling relations that will be studied in detail in Section 6.

Recall that a surface is C^k-continuous if it has a regular (with non-zero Jacobian) C^k local parameterization for each point. A C^1 continuous surface is called *smooth.*

It would be tempting to define a C^k-continuous subdivision scheme as a scheme that always produces C^k-continuous surfaces. Unfortunately, this is too restrictive for stationary subdivision: it appears to be impossible to construct a useful scheme of smoothness higher then C^1 satisfying this requirement. If the definition of C^k-continuity is relaxed to allow the surface to be non-smooth for a proper subspace of the space of all initial control points, then the class of smooth subdivision schemes is sufficiently rich for useful constructions.

Definition 5.1. *A subdivision scheme is called C^k-continuous on a complex K if the following two conditions are satisfied*

- *For all initial values $p \in \mathcal{P}(V, \mathbf{R}^3)$, where V is the vertex set of the complex K, the limit functions $f[p]$ are smooth, except for p from a proper subspace of $\mathcal{P}(V, \mathbf{R}^3)$.*
- *When $f[p]$ is smooth, it is C^k-continuous.*

Due to local definition and compact support properties, we can consider only complexes with finite V (finite subcomplexes of k-regular complexes). In this case, $\mathcal{P}(V, \mathbf{R}^3)$ is a finite-dimensional space which can be identified with a Euclidean space. Any proper subspace of $\mathcal{P}(V, \mathbf{R}^3)$ has measure zero. If we want to prove that a scheme is not smooth on a complex, it is sufficient to show that for a subset $A \in \mathcal{P}(V, \mathbf{R}^3)$ the limit surfaces for all $p \in A$ are not smooth and that the measure of A is not zero. The latter can be shown by demonstrating that A contains a non-empty open set. We will use this approach to prove necessary conditions for smoothness.

To show that a limit surface is smooth around an extraordinary vertex, we need to find a smooth regular parameterization around that vertex. Although the subdivision surface has natural local parameterizations by construction, this parameterization typically is not smooth.

The key observation for finding parameterizations of the surface can be made if we consider Equation 4.5: as the subdivision level increases, in general, the control set for the patch around the extraordinary vertex approaches the control set which is a linear combination of a pair of eigenvectors corresponding to the eigenvalues with largest magnitudes. Otherwise, the tangent plane to the surface at that point would be defined by the control set which is a linear combination of the two eigenvectors corresponding to the largest eigenvalues.

Thus, there is a map such that almost any surface generated by the subdivision scheme, "looks similar" to this map near an extraordinary vertex. This map is completely determined by the leading eigenvectors of the subdivision matrix and subdivision rules for the regular complex. We will call this map characteristic, as it is similar in spirit to the Reif's construction.

We restrict the class of subdivision schemes that we consider by assuming non-degeneracy and stability conditions (Section 5.1).

Under these conditions, we formulate necessary and sufficient criteria for geometric smoothness (existence of limit normals) and C^k continuity in terms of the limit functions generated by the eigenvectors of the subdivision matrix and the characteristic map.

The general conditions for C^k-continuity are rather complicated; if the leading eigenvalues coincide, the conditions are greatly simplified: it is required that all nontrivial eigenvalues λ with $|\lambda| > |\lambda_a^k|$ are of the form λ_a^j and corresponding eigenbasis functions are polynomials of degree j.

Injectivity of the characteristic map is not a necessary condition for subdivision to be smooth, even with our additional assumptions. It should have a specific structure determined by the fact that its Jacobian, which is defined almost everywhere, cannot be 0. This structure is similar to the structure of analytic functions with a branch point at zero; thus, if it is not injective, it is likely that the subdivision scheme is not useful for practical purposes.

5.1 Nondegeneracy and Stability Conditions.

We will assume two conditions.

Before we formulate the conditions, several important observations can be made about the derivatives of the limit functions on U_1 in the k-regular complex $\mathcal{R}_k$.

It is easy to see that if the subdivision scheme is C^k-continuous on the regular complex, the limit functions are C^k on each triangle $(0, i, i+1)$. Moreover, one-sided partial derivatives are defined on the boundary of each triangle, although the values on the different sides of a shared boundary might differ. The only exception is the central vertex (zero), where the derivatives might not exist. Therefore, we can think of the Jacobian of any limit mapping $U_1 \to \mathbf{R}^2$ as being defined everywhere except 0 on the disjoint union of triangles of U_1.

Condition A (Nondegeneracy). At no non-zero point the system of eigenbasis functions f_{ij} has all pairwise Jacobians $J[f_{ij}, f_{kl}]$ equal to zero simultaneously.

Condition B (Stability). If $p \in \mathcal{P}(N_L)$, $p \neq 0$, then $f[p] \neq 0$.

The origin of the first requirement is the techniques of the proof that we are using. This condition is a rather natural assumption of absence of "very degenerate" points in the parameterization of the surface, but it is likely to be possible to construct examples of smooth schemes when this assumption is violated.

Condition B can be derived from the standard stability condition for subdivision on regular complexes. If a scheme does not satisfy this condition, there may be arbitrarily large initial values that produce zero limit function. This condition is not crucial for us, but it allows to eliminate a significant number of degenerate cases and simplifies exposition. In practice it does not seem to be restrictive.

5.2 Dominant Eigenvalues and Eigenbasis Functions

In this section we will find conditions on the *dominant eigenvalues* and *pairs of dominant eigenbasis functions* of the subdivision matrix that are necessary for geometric smoothness, which is a weaker notion of smoothness, requiring only existence of the limit normal. Informally speaking, a pair of eigenbasis functions is *dominant* if it affects the limit of the normals at 0 for a set of surfaces of positive measure.

We proceed in two steps. First, we show that for existence of the limit normal it is necessary that the Jacobians for any pair of dominant eigenbasis functions satisfy certain scaling relations. Then we show that the pair of dominant eigenfunctions has to be unique.

As a corollary, we derive conditions on the dominant eigenvalues that are necessary for geometric smoothness of subdivision. Together with additional conditions derived in the next section, these conditions will lead us to the necessary and sufficient conditions for smoothness.

We prove necessity in a general setting, using only rather weak assumptions of non-degeneracy and stability.

In the special case of Catmull-Clark subdivision necessary conditions for geometric continuity were studied by Ball and Storry [3].

Geometric smoothness. Consider the limit surface $f[x](y) : U_1 \to \mathbf{R}^3$ corresponding to the initial values $x \in \mathcal{P}(N_L, \mathbf{R}^3)$.

At all points where both partial derivatives are not zero, the unit normal is defined. By using a different parametrization it is easy to show that the normal is defined unambiguously at the boundaries of the triangles of U_1 away from zero.

Recall the standard definition of geometric smoothness:

Definition 5.2. *A surface is geometrically (G^1) smooth at a point p if the limit of the normals exist at p.*

For a scheme to be geometrically smooth, the limit surfaces should be geometrically smooth for all initial values with the exception of a proper subspace of $\mathcal{P}(N_L, \mathbf{R}^3)$. This is a weaker notion of smoothness then the one that was defined in the beginning of Section 5.

Dominant pairs of eigenbasis functions. The limit surface f can be decomposed with respect to the real eigenbasis functions (Section 4):

$$f[x](y) = \sum_{i,j} k_{ij} g_{ij}(y), \quad k_{ij} = [a_{ij}, b_{ij}, c_{ij}]^T$$

In all formulas we will assume summation over all eigenvalues and all eigenvectors for each eigenvalue. If λ_i is a complex eigenvalue, we will use the notation $\bar{i}$ to denote the index of the eigenvalue $\bar{\lambda}_i$.

Consider the vector $v(y) = \partial_1 f(y) \times \partial_2 f(y)$. Whenever it is not zero, the unit normal $v_n(y)$ at the point $y \neq 0$ points along this vector. For the normals to converge to a limit at 0, it is necessary that for any sequences of points y_s^1, y_s^2, $s = 1..$, such that $\lim_{s\to\infty} y_s^1 = \lim_{s\to\infty} y_s^2 = 0$, the sequence of vectors $v_n(y_S^1) \times v_n(y_s^2)$ converges to 0.

The expression for vectors $v(y)$ is

$$v(y) = [\sum_{i,j,k,l} b_{ij} c_{kl} J[g_{ij}, g_{kl}], -\sum_{i,j,k,l} a_{ij} c_{kl} J[g_{ij}, g_{kl}], \sum_{i,j,k,l} a_{ij} b_{kl} J[g_{ij}, g_{kl}]]^T$$

For any $\epsilon > 0$, and any choice of two Jacobians $J[g_{i_0 j_0}, g_{k_0 l_0}]$ and $J[g_{i_1 j_1}, g_{k_1 l_1}]$ we can choose a ball B in $\mathbf{R}^{3N}$, where N is the size of N_L, such that for all x in that ball that $||v(y) - \tilde{v}(y)||/||\tilde{v}(y)|| < \epsilon$, where

$$\tilde{v}(y) = [J[g_{i_0 j_0}, g_{kl}](y), J[g_{i_1 j_1}, g_{k_1 l_1}](y), 0]$$

assuming that this vector is nonzero.

If the same estimate holds for a sequence of y_s, and the directions of $\tilde{v}(y_s)$ does not have a limit at 0, then the same is true for any x in B. Such sequences exist in general only for the *dominant eigenbasis functions* which we will define below. If the vector $\tilde{v}(y_s)$ for a pair of dominant eigenbasis functions does not have a limit direction, the scheme is not geometrically smooth, hence is not smooth.

Any g_{ij} can be written as $(1/2)(f_{ij} + \bar{f_{ij}})$, where f_{ij} is the corresponding complex-valued eigenbasis function. We would like to obtain the expression for $v(y/2^s)$, where s is an integer. Each term $J[f_{ij}, f_{kl}](y/2^s)$ is

$$J[f_{ij}, f_{kl}](y^s) = 4^s \sum_{q,r=0}^{i,j} \binom{s}{q} \binom{s}{r} \lambda_i^{s-q} \lambda_k^{s-r} J[f_{ij-q}, f_{kl-r}](y) \tag{5.1}$$

For the real-valued Jacobian $J[g_{ij}, g_{kl}]$ we get

$$J[g_{ij}, g_{kl}](y^s) = 4^s \sum_{q,r=0}^{i,j} \binom{s}{q} \binom{s}{r} |\lambda_i|^{s-q} |\lambda_k|^{s-r}$$
$$\left(|J[f_{ij-q}, f_{kl-r}](y)| \cos(\phi_1(s-q) + \phi_2(s-r) + \psi^1_{ij-q,kl-r}) + |J[f_{ij-q}, \bar{f}_{kl-r}](y)| \cos(\phi_1(s-q) - \phi_2(s-r) + \psi^2_{ij-q,kl-r})\right) \tag{5}$$

where ϕ_i, ϕ_k are the complex phases of the eigenvalues, $\psi^1_{ij-q,kl-r}$ is the phase of $J[f_{ij-q}, f_{kl-r}](y)$, and $\psi^2_{ij-q,kl-r}$ is the phase of $J[f_{ij-q}, \bar{f}_{kl-r}](y)$.

Changing indices, we obtain

$$J[g_{ij}, g_{kl}]\left(\frac{y}{2^s}\right) = 4^s \sum_{q,r=0}^{j,l} \binom{s}{j-q}\binom{s}{l-r} |\lambda_i|^{s-j+q}|\lambda_k|^{s-l+r}$$
$$\left(|J[f_{iq}, f_{kr}](y)| \cos(\phi_1(s-j+q) + \phi_2(s-l+r) + \psi^1_{iq,kr}) + |J[f_{iq}, \bar{f}_{kr}](y)| \cos(\phi_1(s-j+q) - \phi_2(s-l+r) + \psi^2_{iq,kr})\right) \quad (5.3)$$

The expression for $J[g_{ij}, g_{\bar{k}l}]$, which will also be of interest to us, can be written as

$$J[g_{ij}, g_{\bar{k}l}]\left(\frac{y}{2^s}\right) = 4^s \sum_{q,r=0}^{j,l} \binom{s}{j-q}\binom{s}{l-r} |\lambda_i|^{s-j+q}|\lambda_k|^{s-l+r}$$
$$\left(-|J[f_{iq}, f_{kr}](y)| \sin(\phi_1(s-j+q) + \phi_2(s-l+r) + \psi^1_{iq,kr}) + |J[f_{iq}, \bar{f_{kr}}](y)| \sin(\phi_1(s-j+q) - \phi_2(s-l+r) + \psi^2_{iq,kr})\right) \quad (5.4)$$

We will assume that unless both λ_i and λ_k are real, λ_k is chosen to be complex.

As $s \to \infty$, $\binom{s}{i-q}\binom{s}{j-r}$ increases as $s^{i+j-q-r}$. From 5.2 above we see that the behavior of normal is dominated by non-zero terms with $|\lambda_i\lambda_k|$ maximal and $q+r$ minimal. Note that in the expression above, the dominating term does not depend on the choice of j and l. As $s \to \infty$, all terms $J[g_{ij}, g_{kl}](y/2^s)$, i, k fixed, approach $s^{(i+j)-(q+r)} J[g_{iq}, g_{ir}](y/2^s)$, where $q+r$ is minimal pair such that the $J[g_{iq}, g_{ir}]/|\lambda_i\lambda_k|^s$ does not converge to zero. This observation leads to the following definition

Definition 5.3. *Let Λ be the set of pairs eigenvalues for which there is at least one pair of eigenbasis functions with Jacobian not identically equal to zero. A pair of eigenbasis functions $(g_{ij}, g_{kl}$ is called* dominant *if $|\lambda_i\lambda_k|$ is maximal in Λ, and $(n_i + n_k) - (i+j)$ is maximal for all $\lambda_{i'}, \lambda_{k'}$ with maximal $|\lambda_{i'}\lambda_{k'}|$. The map $(g_{ij}, g_{kl}) : U_1 \to \mathbf{R}^3$ is called the characteristic map.*

Asymptotic behavior of the normal at $y/2^s$ as $s \to \infty$ depends only on the dominant eigenbasis functions. Note that according to our definition the characteristic map does not have to be unique. We will show that for smooth schemes it is in fact unique.

$$\mathcal{B}_{ijq,klr}(s) = |J[f_{iq}, f_{kr}](y)| \cos(\phi_1(s-j+q) + \phi_2(s-l+r) + \psi^1_{iq,kr}) +$$
$$|J[f_{iq}, \bar{f_{kr}}](y)| \cos(\phi_1(s-j+q) - \phi_2(s-l+r) + \psi^2_{iq,kr}) \quad (5.5)$$

Then (5.3) is reduced to

$$J[g_{ij}, g_{kl}]\left(\frac{y}{2^s}\right) = 4^s \sum_{q,r=0}^{j,l} \binom{s}{j-q}\binom{s}{l-r} |\lambda_i|^{s-j+q}|\lambda_k|^{s-l+r} B_{ijq,klr}(s)$$

Rotating normals. Consider $\tilde{v}$ constructed as above, with $(i_0, j_0, k_0, l_0) = (i, j, k, l)$ and $(i_0, j_0, k_0, l_0) = (i, j, \bar{k}, l)$. In this case, $\tilde{v}(y/2^s)$ is

$$\tilde{v}\left(\frac{y}{2^s}\right) = \begin{bmatrix} 4^s \sum_{q,r=0}^{j,l} \binom{s}{j-q}\binom{s}{l-r} |\lambda_i|^{s-j+q}|\lambda_k|^{s-l+r} B_{ijq,klr}(s) \\ 4^s \sum_{q,r=0}^{j,l} \binom{s}{j-q}\binom{s}{l-r} |\lambda_i|^{s-j+q}|\lambda_k|^{s-l+r} B_{ijq,\bar{k}lr}(s) \\ 0 \end{bmatrix}$$

Asymptotically, the direction of this vector approaches the direction of

$$w(s) = [B_{ijq,klr}(s), B_{ijq,\bar{k}lr}(s), 0]^T$$

For fixed i, j, k, l, introduce the following notation

$$
\begin{aligned}
A &= J[f_{iq}, f_{kr}],\\
B &= J[f_{iq}, \bar{f}_{kr}],\\
\omega_1 &= \phi_i + \phi_k,\\
\omega_2 &= \phi_i - \phi_k,\\
\zeta_1 &= \phi_1(q-j) + \phi_2(r-l) + \psi^1_{iq,kr},\\
\zeta_2 &= \phi_1(q-j) - \phi_2(r-l) + \psi^2_{iq,kr},
\end{aligned}
$$

Then the vector $w(s)$ is

$$
w(s) = \begin{bmatrix} A\cos 2\pi(\omega_1 s + \zeta_1) + B\cos 2\pi(\omega_2 s + \zeta_2) \\ -A\sin 2\pi(\omega_1 s + \zeta_1) + B\sin 2\pi(\omega_1 s + \zeta_1) \\ 0 \end{bmatrix} \tag{5.6}
$$

Consider $w(s_1)\times w(s_2)$. If there is $C > 0$ such that for any s there is $s_1, s_2 > s$, $||w(s_1)\times w(s_2)|| > C$, then the directions of $w(s)$ don't converge to a limit, and there is no limit normal at 0.

The following lemma answers the question when $w(s)$ converges to a limit.

Lemma 5.1. *The sequence of vectors $w(s)$ defined by* (5.6) *has a limit direction if and only if one of the following is true:*

1. $A = 0$ *and* $\omega_2 = 0$
2. $B = 0$ *and* $\omega_1 = 0$
3. $A = B = 0$
4. $\omega_1 = \omega_2 = 0$
5. $A^2 - B^2 = 0$ *and* $\omega_1 = \omega_2$.

Proof. Without the loss of generality we can assume that ω_1 and ω_2 are less then $1/2$ (we allow A and B to be have arbitrary sign.)

We will consider several cases.

1. ω_1, ω_2, 1 are linearly independent over $\mathbf{Z}$.

   By Kronecker's theorem (see, for example, Hardy [12]), for any ϵ we can find a sequence s_i such that there are sequences of integers n_i, m_i for which $|\omega_1 s_i + \zeta_1 - n_i| \le \epsilon$, $|\omega_2 s_i + \zeta_2 - 1/4 - m_i| \le \epsilon$. We can also find another sequence s'_i, such that there are sequences n'_i and m'_i for which $|\omega_1 s_i + \zeta_1 - 3/4 - n'_i| \le \epsilon$, $|\omega_2 s_i + \zeta_2 - 1/2 - m'_i| \le \epsilon$. Thus, $w(s_i)$ can be made arbitrarily close to $(A, B)^T$ and $w(s'_i)$ can be made arbitrarily close to $(-B, A)^T$. If $\lim_{i\to\infty} w(s_i)\times w(s'_i) = 0$, then $A^2 + B^2 = 0$, which means that both A and B have to be 0.

2. ω_1, 1 and ω_2,1 are linearly independent over $\mathbf{Z}$, ω_1 and ω_2 are linearly dependent over $\mathbf{Z}$.

   Suppose $q\omega_1 - r\omega_2 = 0$. For any p, $w(t)\times w(t+p)$, where t is a real and p is an integer, is periodic with an irrational period $T = LCD(p,q)/r\omega_1$.

   Suppose for some real t and integer p $w(t)\times w(t+p) \ne 0$. Again, by Kronecker's Theorem, we can choose an infinite sequence s_i such that for any i there is an integer k satisfying $|s_i - t - kT| < \epsilon$. For sufficiently small ϵ, $w(s_i)\times w(s_i+1) \ne 0$. We conclude that $w(t)\times w(t+p) = 0$ for all real t and integer p.

   Writing the expression for $w(t)\times w(t+p)$ explicitly, we obtain

$$
\begin{aligned}
&- A^2 \sin 2\pi\omega_1 p + B^2 \sin 2\pi\omega_2 p \\
&+ AB \left(\sin 2\pi \left((\omega_2+\omega_1)t + (\zeta_1+\zeta_2) + \omega_2 p\right) - \sin 2\pi \left((\omega_2+\omega_1)t + (\zeta_1+\zeta_2) + \omega_1 p\right)\right)
\end{aligned} \tag{5.7}
$$

By changing variables and denoting $\omega_1 + \omega_2 = \Omega$, we get

$$
\begin{aligned}
&- A^2 \sin 2\pi\omega_1 p + B^2 \sin 2\pi\omega_2 p \\
&\qquad + AB \left(\sin 2\pi\Omega t(\cos 2\pi\omega_2 p - \cos 2\pi\omega_1 p) + \cos 2\pi\Omega t(\sin 2\pi\omega_2 p - \sin 2\pi\omega_1 p)\right)
\end{aligned} \tag{5.8}
$$

If $\Omega \neq 1$, this expression is identically equal to zero for all t only if each of the following expressions is zero:

$$
\begin{aligned}
-A^2 \sin 2\pi\omega_1 p + B^2 \sin 2\pi\omega_2 p &= 0 \\
AB(\cos 2\pi\omega_2 p - \cos 2\pi\omega_1 p) &= 0 \\
AB(\sin 2\pi\omega_2 p - \sin 2\pi\omega_1 p) &= 0
\end{aligned}
$$

if $A \neq 0$ and $B \neq 0$, from the last two equations for $p = 1$ it follows that $\omega_1 = \omega_2$. Then from the first equation we get $A^2 - B^2 = 0$.

It follows from the first equation that if $A = 0$, ω_2 should be zero (it cannot be 1/2 by assumption). If $B = 0$, $\omega_1 = 0$; both cases are impossible by assumption.

If $\Omega = 1$, then we get only the first equation and for $p = 1$ it can be written as $(A^2 + B^2) sin 2\pi\omega_1 = 0$, i.e. $A = B = 0$.

3. ω_1, ω_2 and linearly independent over $\mathbf{Z}$, ω_1 and 1 are linearly dependent over $\mathbf{Z}$.

   This case is similar to the previous one. ω_1 and 1 are linearly dependent if and only if ω_1 is rational. Suppose $\omega_1 = p/q$. Replacing t with qt we get

$$
\begin{aligned}
&w(qt) \times w(qt + p) = \\
&\quad AB \left(\sin 2\pi(\omega_2 qt + \omega_2 p + \zeta_1 + \zeta_2) - \sin 2\pi(\omega_2 qt + \omega_1 p + \zeta_1 + \zeta_2)\right) + B^2 \sin 2\pi\omega_2 p
\end{aligned} \tag{5.9}
$$

   Similar to the previous case, this expression should be zero for all real t. As $\omega_2 \neq 0$, in a similar way we obtain the system

$$
\begin{aligned}
B^2 \sin 2\pi(\omega_2 p) &= 0 \\
AB(\cos 2\pi\omega_2 p - \cos 2\pi\omega_1 p) &= 0 \\
AB(\sin 2\pi\omega_2 p - \sin 2\pi\omega_1 p) &= 0
\end{aligned}
$$

   As ω_1 cannot be equal to ω_2, either and $\omega_2 \neq 0$, B has to be 0. Therefore, $w(s_1) \times w(s_2)$ is simply $A^2 \sin 2\pi\omega_1(s_1 - s_2)$. Unless $A = 0$, this means that $\omega_1 = 0$.

4. ω_1, ω_2 and 1 are linearly dependent over $\mathbf{Z}$. In this case both ω_1 and ω_2 are rational, and the function $w(s) \times w(s + p)$ is periodic for all p. The only way for it to converge to zero as $s \to \infty$ is to be equal to zero for all p and s.

   Suppose $\omega_1 = r_1/q$, $\omega_2 = r_2/q$. Denote $\omega = 1/q$. Then $w(s) \times w(s + p)$ can be written as

$$B^2 \sin 2\pi\omega r_2 p - A^2 2\pi\omega r_1 p + 2AB \sin 2\pi\omega \frac{r_2 - r_1}{2} p \cos 2\pi(\omega k(s + \frac{p}{2}) + \zeta) = 0 \qquad (5.10)$$

where $k = r1 + r2$, $\zeta = \zeta_1 + \zeta_2$.

Assume $A \neq 0$ and $B \neq 0$. The equation has to be true for any choice of p and s. Adding equations for $2p$ and $q - 2p$ we get

$$4AB \sin 2\pi\omega \frac{r_2 - r_1}{2} p \sin 2\pi\omega ks \sin 2\pi\omega kp = 0$$

As $\omega_1 + \omega_2 < 1$, $k\omega < 1$; therefore it should be $1/2$, unless $r_2 = r_1$. A check shows that $1/2$ still is not a solution for odd p, unless $r_2 = r_1$, i.e. $\omega_1 = \omega_2$. In this case, $A^2 - B^2$ also has to be zero.

Suppose $A = 0$. Then the equation is reduced to $B^2 \sin 2\pi\omega r_2 p = 0$, which requires $\omega_2 =$ or $B = 0$. Similarly, $B = 0$ implies $A = 0$ or $\omega_1 = 0$.

□

Dominant pairs with alternating Jacobian. It follows from Lemma 5.1 that a dominant pair of eigenbasis functions may correspond only to a pair of real eigenvalues ($\omega_1 = \omega_2 = 0$), to a pair of eigenvalues with the same phase ($\omega_2 = 0, A = 0$) or to a pair of eigenvalues with opposite phases $\omega_1 = 0, B = 0$. The case $\omega_1 = \omega_2$ would imply that $\phi_k = 0$, which means that ϕ_i also should be zero (we have assumed that λ_k is real only when both λ_i and λ_k are real).

In all possible cases, The Jacobian of the pair satisfies the scaling relation

$$J[g_{iq}, g_{kr}]\left(\frac{y}{2^s}\right) = 4^s(\lambda_i\lambda_k)^s[g_{iq}, g_{kr}](y)$$

This is an important fact that allows us to prove the following Lemma.

Lemma 5.2. *If the subdivision scheme is smooth on a k-complex, and satisfies Condition A, any characteristic map $\Phi = (g_{ab}, g_{cd})$ for this complex is a covering projection of $\dot{U}_1$ onto $\Phi(\dot{U}_1)$ and its Jacobian cannot be zero anywhere.*

Sketch of the proof. First, we show that if the Jacobian is zero at some point, we can construct a set of initial control points $p \in \mathcal{P}(V, \mathbf{R}^2)$ in the plane such that the Jacobian of the limit mapping corresponding to these control points changes sign arbitrarily close to 0. Any small perturbation of this set also produces a mapping with similar properties.

Then we prove that for any such mapping from U_1 to the plane we can construct a set of control points $p' \in \mathcal{P}(V, \mathbf{R}^3)$ such that the limit surface corresponding to this set will not be geometrically smooth, and this property is preserved under arbitrary small perturbations (Figure 5.2). This means that the scheme is not smooth.

Finally, we conclude that if the Jacobian is not zero anywhere, then for any point of $\dot{U}_1$ the local smooth parameterization Φ' mentioned above is regular and is, therefore, a homeomorphism. This proves that Φ is a covering projection away from 0.

Limit mappings with alternating Jacobian. As we have seen, the Jacobian of the mapping Φ is defined everywhere on $\dot{U}_1$ (two values may be defined for the boundaries between triangles of U_1).

As we have shown,

$$J[\Phi]\left(\frac{y^1}{2^n}\right) = 4^n(\lambda_a\lambda_c)^n J[\Phi](y^1)$$

At a point y,

$$\Phi(y) = (\Phi_1(y), \Phi_2(y)) = \left(\sum_v x_{ab}(v)\phi_v(y), \sum_v x_{cd}(v)\phi_v(y)\right) \tag{5.11}$$

where summation is over the vertices in support of the eigenvectors; ϕ_v are the basis functions defined in Theorem 3.1.

Denote $\boldsymbol{\phi}(y)$ the vector of $\phi_v(y)$ for all v. Then the expression for Φ is

$$\Phi(y) = ((x_1, \boldsymbol{\phi}(y)), (x_2, \boldsymbol{\phi}(y)))$$

The Jacobian of the mapping can be written in the following form (one-sided derivatives are implied on the boundaries between triangles).

$$J[\Phi] = (x_1, (x_2, \partial_1\boldsymbol{\phi})\partial_2\boldsymbol{\phi} - (x_2, \partial_2\boldsymbol{\phi})\partial_1\boldsymbol{\phi}) \tag{5.12}$$

Suppose the Jacobian is 0 for some point of $\dot{U}$, which we will denote by y^0. It follows from the scaling relations that $J[\Phi](y^0/2^n) = 0$. Denote

$$v_n = \left(x_2, \partial_1\boldsymbol{\phi}\left(\frac{y^0}{2^n}\right)\right)\partial_2\boldsymbol{\phi}\left(\frac{y^0}{2^n}\right) - \left(x_2, \partial_2\boldsymbol{\phi}\left(\frac{y^0}{2^n}\right)\right)\partial_1\boldsymbol{\phi}\left(\frac{y^0}{2^n}\right)$$

Then the condition $J[\Phi](y^0/2^n) = 0$ can be written as

$$(x_1, v_n) = 0, \quad \text{for all } n. \tag{5.13}$$

As $x_1 \neq 0$ by assumption, the vectors v_n span a proper subspace of R^N, where N is the number of components of x_1. Consider the set of directions of v_n which can be represented as points on an $N-1$ dimensional sphere. As the sphere is compact, the set has a limit point, which we will denote by u. Then for all vectors w satisfying $(w, u) < 0$ ($(w, u) > 0$) for infinitely many vectors v_n the scalar products (w, v_n) will be positive (negative).

Suppose the Jacobian is zero at some point of a given triangle if and only if it is zero at any point of that triangle; in this case, it immediately follows from the observation in the beginning of this section, that the Jacobian is 0 everywhere on U_1, which contradicts the assumptions of the Lemma.

Therefore, there is a point y^1 on the same triangle with y^0 where the Jacobian is not zero. Without the loss of generality we can assume that it is positive there.

Note that if the Jacobian is positive at some point y^1, then it is also positive at all points $y^1/2^{2n}$.

Consider a small perturbation Δx of x_1, such that $(\Delta x, u) < 0$. Then the Jacobian $J[\Phi_{\Delta x}](y^0/2^n)$ is negative for infinitely many values of n, where $\Phi_{\Delta x} = (f[x_1 + \Delta x], f[x_2])$. $J[\Phi_{\Delta x}](y^1/2^n)$ remains positive for infinitely many values of n for sufficiently small magnitude of Δx (say, in L_∞-norm). A straightforward check shows that the signs of Jacobians $J[\Phi_{\Delta x}](y^0/2^n)$ and $J[\Phi_{\Delta x}](y^1/2^n)$ are preserved for any sufficiently small perturbation of $(x_1 + \Delta x, x_2)$: suppose $w^1 + \Delta x = \sum_{ij} a^1_{ij} r_{ij}$, $w^2 = \sum_{ij} a^2_{ij} r_{ij}$. Then

$$J[\Phi_{w+\Delta x}](y) = J[\Phi](y) + \sum_{i,j,k,l} a^1_{ij} a^2_{kl} J[g_{ij}, g_{kl}](y) \tag{5.14}$$

where $\Phi_{w+\Delta x}(y) = (f[x_1\Delta x + w^1](y), f[x_2 + w^2](y))$. The summation is over all pairs of eigenbasis functions different from g_{ab}, g_{cd}.

By definition of the characteristic map, for sufficiently small coefficients a^{ij}, $(i,j) \neq (a,b), (c,d)$, the ratio of the values of $J[\Phi_{w+\Delta x}](y/2^n)$ to $J[\Phi](y/2^n)$ can be made arbitrarily small, if it is not zero.

Thus, for small perturbations $J[\Phi_{w+\Delta x}](y/2^n) > 0$ for infinitely many n.

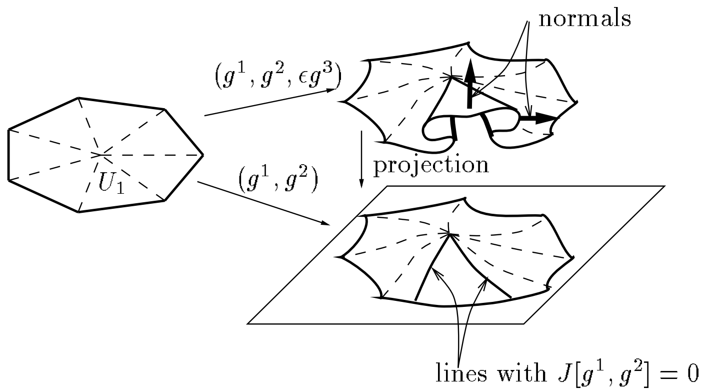

Figure 18: Construction of geometrically non-smooth map.

Construction of non-smooth surfaces. Now we are ready to construct sets of control points for which $\mathbf{R}^2 \to \mathbf{R}^3$ is not geometrically smooth at 0, i.e. the limit of normals does not exist at 0.

Consider a map $(g^1, g^2) : U_1 \to \mathbf{R}^2$ for which the Jacobian changes sign arbitrarily close to 0. As the Jacobian is continuous, there are points arbitrarily close to 0 where it is 0.

Let $y_n^0 = y^0$ and y_n^1, $n = 1 \ldots$, be the sequences of points where the Jacobian is 0 and positive respectively. We will assume that the positions of points satisfy

$$\left\| y_n^i \right\|_2 \leq \left\| y_0^i / 2^{k_n^i} \right\|_2, \quad i = 0, 1$$

were k_n^i are sequences that increase as $n \to \infty$.

Consider the map $F : U_1 :\to \mathbf{R}^3 = (g^1, g^2, \epsilon g^3)$, where g^3 is the limit function corresponding to some arbitrary initial values, ϵ is a small positive number (Figure 5.2).

By condition A, for any point where the Jacobian is 0, we can choose the limit function g^3 such that one of the Jacobians $J[(g^3, g^1), J[g^3, g^2)]$ is not zero. In addition, g^3 can be chosen in such a way that the coefficients of $f[x_1]$ and $f[x_2]$ in its eigenbasis decomposition are not 0.

The expression for the unnormalized normal vector at a point y is

$$\mathbf{n}(y) = \left(\epsilon J[g^2, g^3], \epsilon J[g^3, g^1], J[g^1, g^2]\right)^T = (\epsilon J_{23}, \epsilon J_{31}, J_{12})^T$$

At points y_n^0 the last component is zero, at points y_n^1 it is not zero. Our choice of g^3 ensures that vectors $\mathbf{n}(y_0)$ and $\mathbf{n}(y_1)$ are not zero vectors.

The cosine of the angle between $\mathbf{n}(y_n^1)$ and $\mathbf{n}(y_m^0)$ is given by

$$\cos(\widehat{\mathbf{n}(y_n^0), \mathbf{n}}(y_m^1)) = \frac{\epsilon \left| J_{23}(y_n^0) J_{23}(y_m^1) + J_{13}(y_n^0) J_{13}(y_m^1) \right|}{\sqrt{(J_{12}(y_n^0)^2 + J_{23}(y_n^0)^2 + J_{13}(y_n^0)^2)(J_{23}(y_m^1)^2 + J_{13}(y_m^1)^2)}} \tag{5.15}$$

Using the definition of the characteristic maps and the fact that decomposition of all g^i can be assumed to contain terms f_{ab} and f_{cd} with non-zero coefficients, the ratio

$$\frac{J_{ij}(y_n^i)}{4^n \lambda_a^{k_n} \lambda_b^{k_n} Jij(y_0^1)}$$

can be made arbitrarily close to 1.

Therefore,

$$\left| \frac{J_{ij}(y_n^l)}{4^n \lambda_a^{k_n} \lambda_b^{k_n} J_{ij}(y_0^l)} - 1 \right| < \epsilon', \quad l = 0, 1$$

Thus, we conclude that

$$\left|\cos(\widehat{\mathbf{n}(y_n^0), \mathbf{n}}(y_m^1))\right| =< \epsilon\frac{(1+\epsilon')^2}{(1-\epsilon')^2}\left|\cos(\widehat{\mathbf{n}(y_0^0), \mathbf{n}}(y_0^1))\right| \tag{5.16}$$

for sufficiently large n, m. For sufficiently small ϵ, $\left|\cos(\widehat{\mathbf{n}(y_0^0), \mathbf{n}}(y_0^1))\right|$ can be made arbitrarily small. Therefore, the angle between unit normal vectors at point y_n^1 and y_m^0 can be made greater then $\alpha < \pi/2$ and less then $\pi - \alpha$. The normals cannot have a limit at zero and the surface is not geometrically smooth. □

Uniqueness of the dominant pair Finally, we prove the main result of this section:

Theorem 5.1. *For a geometrically smooth scheme, the characteristic map is unique*

Proof. To simplify the notation, in this proof we replace the double indices of the eigenbasis functions with multiindices: $\alpha = (i_\alpha, j_\alpha)$, $\beta = (i_\beta, j_\beta)$ etc. We will also use use notation $J_{\alpha\beta}$ for $J[g_\alpha, g_\beta]$.

Suppose g_α, g_β and g_γ, g_δ are are two pairs of dominant eigenbasis functions. Assume $|\lambda_{i_\alpha}| \geq |\lambda_{i_\beta}|$ and $|\lambda_{i_\gamma}| \geq |\lambda_{i_\delta}|$.

As we have observed, for smoothness it is necessary that the direction of the vector $w(y) = [J_{\alpha\beta}, J_{\gamma\delta}]^T(y)$ has a limit as $y \to 0$.

By Lemmas 5.1, all vectors $[J_{\alpha\beta}, J_{\gamma\delta}]^T(y/2^s)$ should have the same direction for the scheme to be geometrically smooth.

Thus, if two vectors $w(y_1)$ and $w(y_2)$ have different directions, there are vectors with the same difference in directions arbitrarily close to $y = 0$, and the scheme is not smooth. We conclude that for any choice of $g_\alpha, g_\beta, g_\gamma, g_\delta$ the direction of w should be the same.

Fix a pair of dominant eigenbasis functions g_α, g_β. By Lemma 5.2, the Jacobian of the pair is not zero anywhere.

Suppose $|\lambda_{i_\gamma}| > |\lambda_{i_\alpha}|$. Then necessarily $J_{i_\gamma i_\alpha} = 0$ and $J_{i_\gamma i_\beta} = 0$ everywhere on U_1. These two equations have unique solution with respect to $\partial_1 g_\gamma$ and $\partial_2 g_\gamma$ for any point, because $J_{\alpha\beta} \neq 0$ anywhere. The unique solution is $\partial_1 g_\gamma \equiv 0$, $\partial_2 g_\gamma \equiv 0$; therefore, g_γ is a constant, which is impossible.

In a similar way we can show that $n_{i_\alpha} - n_{j_\alpha} = n_{i_\gamma} - n_{j_\gamma}$.

Thus, for the scheme to be smooth g_γ, g_α, g_γ, g_β necessarily have to be either identically zero, or a dominant pair of eigenbasis functions.

Thus, the direction of the vectors $[J_{\gamma\alpha}, J_{\alpha\beta}, 0](y)$, $[J_{\gamma\beta}, J_{\alpha\beta}, 0](y)$ should be constant for all y. Let y_1 be a fixed point in U_1. Then for any y we have

$$\begin{aligned} J_{\gamma\alpha}(y) &= J_{\alpha\beta}(y)J_{\gamma\alpha}(y_1)/J_{\alpha\beta}(y_1) \\ J_{\gamma\beta}(y) &= J_{\alpha\beta}(y)J_{\gamma\beta}(y_1)/J_{\alpha\beta}(y_1) \end{aligned} \tag{5.17}$$

As $J_{\alpha\beta}(y)$ is not zero anywhere by Lemma 5.2, This system always has a unique solution with respect to $\partial_1 g_\gamma$ and $\partial_2 g_\gamma$. From the form of this solution it easily follows that the function g_γ has to be a linear combination of g_α and g_β. As all eigenbasis vectors are linearly independent, it follows from stability condition that no non-trivial linear combination of eigenbasis functions can be identically equal to zero. We conclude that there can be no dominant pairs of eigenbasis functions different from g_α, g_β.

□

Conditions on dominant eigenvalues. From the Theorem 5.1 and Lemma 5.2 we can easily derive the following necessary and sufficient conditions for geometric smoothness:

Corollary 5.1. *A nondegenerate stable subdivision scheme is geometrically smooth on the k-regular complex if and only if it has a unique characteristic map* $\Phi = (g_{ab}, g_{cd})$ *with Jacobian not equal to zero anywhere, and dominant eigenvalues and eigenbasis functions satisfy one of the following conditions:*

1. $|\lambda_1| = |\lambda_2| > |\lambda_3|$, *$\lambda_1$ and λ_2 either both real or complex conjugate,* $g_{ab} = g_{10}$ *and* $g_{cd} = g_{20}$.

2. $|\lambda_1| > |\lambda_2|$, *both* $\lambda_a = \lambda_1$ *and* λ_c *are real,* $g_{ab} = g_{10}$, *all* g_{ij} *with* $|\lambda_i| \geq |\lambda_c|$ *satisfy* $J[g_{ij}, g_{10}] = 0$.

Proof. *Sufficiency* of these conditions is straightforward — in fact, uniqueness and non-zero Jacobian alone guarantee that the limit normal exists.
Necessity. Suppose there is an eigenbasis function g_{i0} such that $|\lambda_i| \geq |\lambda_b|$, $\lambda_i \neq 1$, $x_{i0} \neq x_{ab}, x_{cd}$.

Then by definition of dominant pairs and uniqueness of characteristic map $J[g_{i0}, g_{ab}] = 0$, $J[g_{i0}, g_{cd}] = 0$. As $J[g_{ab}, g_{cd}] \neq 0$, both partial derivatives of g_{ij} have to be zero, i.e. f_{ij} should be a constant, which is impossible, as $\lambda_i \neq 1$. Thus, a always is 1, up to rearrangement of eigenvalues with maximal magnitude. Similarly, b can always be chosen to be 0.

If λ_1 is real, by Lemma 5.1 λ_c is also real.

If $|\lambda_2| = |\lambda_1|$, c cannot be 1, and x_{20} has to be x_{cd}. In this case $|\lambda_3|$ necessarily has to be less then $|\lambda_1| = |\lambda_2|$. λ_1 and λ_2 can be either real or complex conjugate; in the latter case the corresponding real eigenbasis functions are the real and imaginary part of the complex eigenbasis function of λ_1.

If $|\lambda_1| > |\lambda_2|$, it has to be real: otherwise there would be a distinct complex conjugate eigenvalue of the same magnitude. In this case c can be any index starting with 1. λ_c also has to be real by Lemma 5.1. All eigenbasis functions g_{ij} for which $|\lambda_i| \geq |\lambda_c|$ should satisfy $J[g_{ij}, g_{10}] = 0$. also has to be real, as by Lemma 5.1 the complex phases of the dominant eigenvalues should be equal or opposite. We have shown that $b = 0$, and if $c = 1$, $|\lambda_2| < |\lambda_1|$.

□

5.3 Structure of Eigenbasis Functions

If the characteristic map is a covering, near zero it can be thought of as consisting of several "sheets". This can be seen in the following way. Suppose $\gamma : [0, 1] \to \mathbf{R}^2$ is a curve connecting a non-zero boundary point of $\Phi(\dot{U})$ with zero. Then the set $A = \Phi(\dot{U}) \setminus \gamma([0, 1])$ is simply connected. $\tilde{A} = \dot{U}_1 \setminus \Phi^{-1}(\gamma([0, 1]))$ is a covering of A. It follows from the lifting theorem in algebraic topology (see, for example, Spanier [22]) that the set $\tilde{A}$ can be represented as a disjoint union of sets D_i such that Φ on each set is a homeomorphism onto A. If there is only one set D_i, then the map Φ is injective. This is likely to be the case for useful subdivision schemes, but it is not strictly necessary for smoothness. A model example of a non-injective map which is a covering on U_1 is $z^n : \mathbf{C} \to \mathbf{C}$, with standard identification of $\mathbf{C}$ and $\mathbf{R}^2$. We can construct a map from $\mathbf{R}^2$ to $\mathbf{R}^3$, by taking the first two coordinate functions to be the coordinate functions of z^n considered as a mapping to the plane, and an arbitrary "generic" function as the third coordinate. For a "generic" function, the sheets of z^n will become separated and the surface will have rather complicated structure in the neighborhood of zero, which does not admit a regular parameterization (Figure 5.3).

But our choice of functions is limited to the functions that can be constructed by subdivision, and under certain conditions no "generic" functions are in this set. We will make this construction precise in the following lemma.

We prove the following general lemma.

Lemma 5.3. *Suppose a continuous mapping* $\tilde{\Phi} = (g^1, g^2) : B \to \mathbf{R}^2$ *is a covering on* $\dot{B}$, *where* B *is a closed planar disk,* $\dot{B}$ *is the same disk without the origin. Then for a continuous mapping* $F : \dot{B} \to \mathbf{R}^3$ *defined as* (g^1, g^2, g^3) *the image* $F(B)$ *is homeomorphic to a disk if and only if* $\tilde{\Phi}(t_1) = \tilde{\Phi}(t_2)$ *implies* $g^3(t_1) = g^3(t_2)$.

Proof. The conditions of the lemma are clearly sufficient for the image $F(B)$ to be homeomorphic to a disk.

The idea of the proof of necessity is to construct a closed simple path in the surface that does not separate the surface into two disjoint parts. Clearly, in this case the surface cannot be homeomorphic to a disk.

We will assume that $F(0) = 0$. Note that the number of sets D_i constructed using an incision γ, should be finite, because B is compact. Indeed, suppose a point $\xi \neq 0$ in $\Phi(\dot{B})$ has infinite set $\Phi^{-1}(\xi)$. Then there is a limit point y' of the set which is contained in B. At y' the value of Φ has to

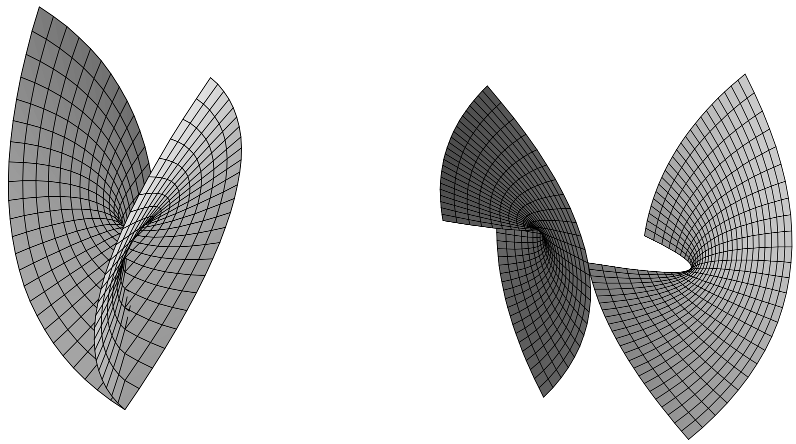

Figure 19: Right: parametric surface $(s^2 - t^2, 2st, s + t)$. The complex form of $(s,t) \to (s^2 - t^2, 2st)$ is $z \to z^2$. The surface does not have a regular parameterization around 0. Left: two sheets of the surface with incision along the ray $s = 0$. One of the sheets is displaced.

be y by continuity, therefore, y' cannot be 0. But clearly this point does not have a neighborhood where the mapping is a homeomorphism, which contradicts the fact that Φ is a covering. Let the number of sheets D_i be n.

Suppose that for some points y_1 and y_2, such that $\tilde{\Phi}(y^1) = \tilde{\Phi}(y_2)$, $g^3(y_1) \neq g^3(y^2)$. We show that there is another pair of points $\tilde{y}^1$ and $\tilde{y}^2$ such that $\tilde{\Phi}(\tilde{y}^1) = \tilde{\Phi}(\tilde{y}^2) = \tilde{\xi}$ and $g^3(\tilde{y}^1) = g^3(\tilde{y}^2)$, i.e. the surface has a self-intersection (Figure 5.3). Consider a simple closed path $\alpha : [0,1] \to \tilde{\Phi}(B)$ surrounding the origin. Such a path exists, because the origin cannot be a boundary point of $\tilde{\Phi}(\dot{B})$. If it were a boundary point, $\tilde{\Phi}(\dot{B})$ would be a disjoint union of simply connected sets or just one simply connected set, and its covering $\dot{B}$ should have been a union of disjoint simply connected sets which is impossible.

Consider a path α_n obtained by going around α n times.

This path can be lifted to a path in $\dot{B}$ that contains a closed subpath $\tilde{\alpha}$. Indeed, the inverse image of any point of the path can contain no more then n points. Therefore, there are distinct parameter values t_1 and t_2 for which the values of the lifted path coincide. Choosing t_2 to be the minimal value of parameter for which the value of the path coincides with a previous value, we obtain a simple closed path $\tilde{\alpha}$ that covers the path α.

The path $\tilde{\alpha}$ goes around zero, otherwise, we would have a null-homotopic path mapped to a path which is not null-homotopic. It cannot be homeomorphic to α as it contains points on several sheets.

Suppose for parameter values t_1, t_2 on $\tilde{\alpha}$ $\tilde{\Phi}(\tilde{\alpha}(t_1)) = \tilde{\Phi}(\tilde{\alpha}(t_2)) = \alpha(t)$ and $g^3(\tilde{\alpha}(t_1)) > g^3(\tilde{\alpha}(t_2))$. Consider the set of points $\tilde{\Phi}^{-1}(\alpha(t)) = \{\tilde{\alpha}(t_1), \tilde{\alpha}(t_2), \dots \tilde{\alpha}(t_k)\}$. where k is the number of sheets of $\tilde{\alpha}$ considered as a covering of α. Without the loss of generality, assume $0 = t_1 < t_2 < \ldots < t_k = 1$. On each interval $[t_i, t_{i+1})$ open on one side, the mapping of $\tilde{\alpha}$ to α is one-to-one. Consider the sequence of values $g^3(\tilde{\alpha}(t_k))$. As $g^3(\tilde{\alpha}(t_1)) > g^3(\tilde{\alpha}(t_2))$ and $g^3(\tilde{\alpha}(t_1)) = g^3(\tilde{\alpha}(t_k))$, then there is i such that $g^3(\tilde{\alpha}(t_i)) \geq g^3(\tilde{\alpha}(t_{i-1}))$ and $g^3(\tilde{\alpha}(t_{i+1})) < g^3(\tilde{\alpha}(t_{i-1}))$. Let ω_j be the inverse of $\tilde{\Phi}|_{\tilde{\alpha}([t_j,t_{j+1}))}$. Define $g^3_j : [0,1] \to \mathbf{R}$ as $g^3_i(t) = g^3(\omega_i(\alpha(t))$ for $0 \leq t < 1$, $g^3(\omega_{j+1}(\alpha(0))$ for $t = 1$. Clearly g^3_j are continuous. Consider g^3_{i-1} and g^3_i. As $g^3_i(0) \geq g^3_{i-1}(0)$ and $g^3_i(1) < g^3_{i-1}(1)$, there is a point where the values of g^3_i and g^3_{i-1} coincide. This proves that there is a pair of points $\tilde{y}_1, \tilde{y}_2 \in \dot{B}$ for which $F(\tilde{y}_1) = F(\tilde{y}_2)$.

The points y_1,y_2,$\tilde{y}_1$,$\tilde{y}_2$ can be chosen in such a way that $y_1, \tilde{y}_1 \in D_i$, $y_2, \tilde{y}_2 \in D_j$. Let $\tilde{\xi} = \tilde{\Phi}(\tilde{y}_1)$.

Consider a path β in $\tilde{\Phi}(\dot{B})$ connecting 0 and $\tilde{\xi}$, going through ξ, lying in the interior of $\tilde{\Phi}(\dot{B})$ (Figure 5.3). Suppose the incision γ is chosen in such a way that it does not intersect β except

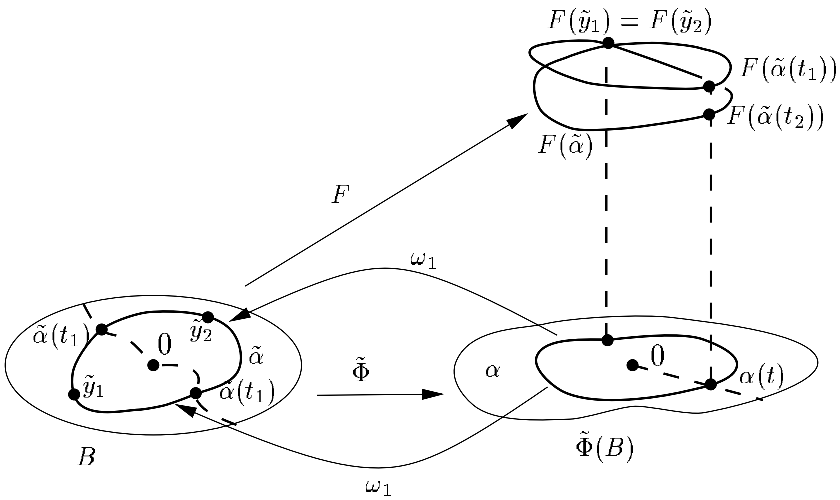

Figure 20: Existence of a self-intersection. The surface itself is not shown; the picture shows the case $n = 2$.

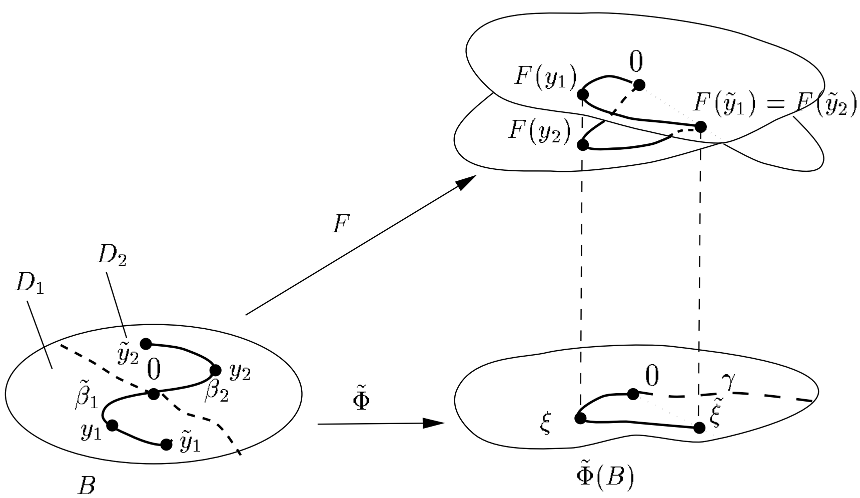

Figure 21: A cut along $F(\tilde{\beta}_1)$ and $F(\tilde{\beta}_2)$ does not separate the surface into two disjoint parts.

at zero. As $\tilde{\Phi}$ is a homeomorphism on each domain D_l, each domain D_i and D_j contains a path which is a homeomorphic image of β without the endpoint 0. Clearly each can be extended to be a homeomorphic image of β by adding zero. Denote these paths $\tilde{\beta}_i$ and $\tilde{\beta}_j$.

Consider the images $F(\tilde{\beta}_i)$ and $F(\tilde{\beta}_j)$. These images are paths in the surface $F(\dot{B}))$. By construction, they both have one-to-one parametrizations over β, as beta is a homeomorphic image of $[0, 1]$, β_i and β_j are parametrized over $[0, 1]$. We know that the curves $F(\tilde{\beta}_i)$ and $F(\tilde{\beta}_j)$ coincide at the endpoints and are different at some point in the middle, corresponding to the value of parameter $t \in [0, 1]$ such that $\beta(t) = \xi$. Let

$$t_0 = \inf \left\{ t \in [0,1] \;\middle|\; g^3(\tilde{\beta}_i(t)) \neq g^3(\tilde{\beta}_j(t)) \right\}$$
$$t_1 = \sup \left\{ t \in [0,1] \;\middle|\; g^3(\tilde{\beta}_i(t)) \neq g^3(\tilde{\beta}_j(t)) \right\}$$

Then by concatenating parts of the curve β_i and β_j over $[t_0, t_1]$ we get a path τ in $F(B)$ homeomorphic to a circle.

The set $B \setminus (\beta_i([0,1]) \cup \beta_j([0,1]))$ is path-connected (the proof of this fact is not entirely trivial, for an elementary proof see, for example, Stillwell [23]).

Therefore, if we remove the simple closed path τ from the surface $F(B)$, it will remain path-connected.

Suppose that there is a homeomorphism mapping $F(\dot{B})$ onto a simply connected part of the plane B. Then any closed path contained in B will separate B into two disconnected components. This contradicts our previous conclusion.

We have constructed a limit surface for which a neighborhood of zero is not homeomorphic to a subset of the plane. Any small perturbation will retain this property, therefore, the subdivision scheme is not smooth.

□

Now we can easily prove the following

Lemma 5.4. *If a subdivision scheme satisfying conditions A and B is smooth on the k-regular complex, and for two points y^1 and y^2 $\Phi(y^1) = \Phi(y^2)$ then for any eigenbasis limit function $f_{ij}(y^1) = f_{ij}(y^2)$; if this condition does not hold, there is an open set of surfaces that are not homeomorphic to a disk in any neighborhood of zero.*

Note. The first part of the lemma can be proved without resorting to the topological constructions of Lemma 5.3, using the fact that if a surface is smooth, it can be parameterized over the tangent plane with parameterization given by projection into the tangent plane. Our proof gives more information about the structure of the surface and proves a stronger fact – the surface does not admit a C^0-parameterization, not only C^1-parameterization.

Proof. Suppose that for some eigenbasis function $f_{ij}(y_1) \neq f_{ij}(y_2)$ for points y_1, y_2 such that $\Phi(y_1) = \Phi(y_2)$. The same would be true for any sufficiently small perturbation $\tilde{\Phi}$ of Φ.

Consider a limit function g and the difference

$$g(y_1) - g(y_2) = \sum_{ij} a_{ij}(f_{ij}(y_1) - f_{ij}(y_2))$$

Let (s,t) be the minimal index for which $a_{st} \neq 0$ and $f_{st}(y_1) \neq f_{st}(y_2)$ for some points y_1, y_2 satisfying $\tilde{\Phi}(y_1) = \tilde{\Phi}(y_2)$. For $g\left(y/2^l\right)$, $l > n_i$ for all i, we have

$$g(\frac{y}{2^l}) = \sum_i \lambda^{l-n_i} \sum_j \sum_{j=0}^{n} f_{ij}(y) \left(\sum_{m=0}^{n-j} \binom{l}{m} \lambda^{n_i - m} a_{i(m+j)} \right) \tag{5.18}$$

For sufficiently large l all terms with $i > k$ become smaller then the terms with $i = k$.

Consider the term of the difference

$$\lambda_i^{l-n_s} \sum_{j=0}^{n} (f_{sj}(y_1) - f_{sj}(y_2)) \left(\sum_{m=0}^{n_s - j} BCml\lambda^{n_s - m} a_{s(m+j)} \right)$$

.

Suppose $a_{sn_s} \neq 0$. For $l \to \infty$ dividing this expression by l_s^n, we can observe that the absolute value of this expression behaves as

$$(1 - \epsilon(l))\lambda_i^{n_s - m} \binom{l}{n-t} a_{in}(f_{st}(y_1) - f_{st}(y_2))$$

where $\epsilon(l)$ is a function approaching 0 as $l \to \infty$.

As all other terms in the sum (5.18) decay faster as $l \to \infty$, we conclude that infinitely many terms of the series $g(y_1/2^l) - g(y_2/2^l)$ are non-zero.

This means that for any neighborhood of zero the conditions of Lemma 5.3 are satisfied, and the part of the surface near zero is not homeomorphic to a disk. As the choice of coefficients a_{ij} was almost arbitrary and so was the choice of $\tilde{\Phi}$, this is true for a set of surfaces of measure greater than zero. □

5.4 Smoothness Criterion

In this section we formulate the criterion for C^k-continuity. In the previous section we have shown that for a subdivision scheme to be smooth it is necessary that whenever $\Phi(y^1) = \Phi(y^2)$, for any eigenbasis function f, $f(y^1) = f(y^2)$. This means that $f(\Phi^{-1}(\xi))$ is unique and the surface can be parameterized over $\Phi(U_1)$.

It is easy to see that the functions $f(\Phi^{-1}(\xi))$ are C^k everywhere except at zero if the scheme is C^k on the regular complex. Indeed, recall that there is an injective mapping μ on a neighborhood A of any point $y \neq 0$ of the k-regular complex that maps it to the regular complex, such that the mapping $f[x] \circ \mu^{-1}$ is a C^k-mapping on A. Writing $f \circ \Phi^{-1}$ as $f \circ \mu^{-1}(\Phi \circ \mu^{-1})^{-1}$ we immediately obtain that the mapping $f \circ \Phi^{-1}$ is C^k-continuous.

Suppose the decomposition of a surface F with respect to the eigenbasis functions is

$$F(y) = \sum_{ij} k_{ij} f[x_{ij}](y)$$

where x_{ij} are the eigenvectors of the subdivision matrix. Assume that the vectors k_{ab} and k_{cd} are independent. The surface can be parameterized as

$$F(\xi) = \sum_{ij} k_{ij} f[x_{ij}](\Phi^{-1}(\xi)) = AP\xi + \sum_{(i,j)\neq(a,b),(c,d)} k_{ij} f[x_{ij}](\Phi^{-1}(\xi))$$

where A is a nondegenerate affine transform, P the projection of ξ into the plane spanned by k_{ab}, k_{cd}.

Lemma 5.4 gives necessary conditions for smoothness. It turns out that under our assumptions these conditions are also sufficient.

Lemma 5.5. *Suppose that for all eigenbasis functions f_{ij} $\Phi(y_1) = \Phi(y_2)$ for $y_1, y_2 \in U_1$ implies $f_{ij}(y_1) = f_{ij}(y_2)$. Then almost all surfaces generated by the subdivision scheme on the k-regular complex are smooth.*

Proof. Consider a surface $F = (g^1(y), g^2(y), g^3(y))$, such that the coefficients k_{ab} and k_{cd} of f_{ab} and f_{cd} in the eigenbasis decomposition of F are not zero.

Then the coordinates in $\mathbf{R}^3$ can be chosen in such a way that k_{ab} and k_{cd} are the first two coordinate vectors. Then g^1 has only f_{ab} with non-zero coefficient in its decomposition, g^2 has only f_{cd} and g^3 has neither. As we have discussed, the surface can be parameterized over the range of Φ. Then $F(\xi)$ can be written as

$$F(\xi) = \begin{pmatrix} \xi_1 + \sum k^1_{ij} f_{ij}(\xi) \\ \xi_2 + \sum k^2_{ij} f_{ij}(\xi) \\ \sum k^3_{ij} f_{ij}(\xi) \end{pmatrix}$$

This parameterization, as it was noted above, is C^k everywhere except 0, if the scheme is C^k on the regular complex.

It follows from the definition of the characteristic map that for all f_{ij}, f_{kl} such that one of them is not a leading eigenvector,

$$\lim_{\|\xi\|_2 \to 0} J[f_{ij}, f_{kl}](\xi) = 0$$

Therefore, the pairwise Jacobians of the coordinate functions have limits at zero and

$$\begin{aligned} \lim_{\|\xi\|_2 \to 0} J[g^1, g^2] &= 1 \\ \lim_{\|\xi\|_2 \to 0} J[g^2, g^3] &= 0 \\ \lim_{\|\xi\|_2 \to 0} J[g^3, g^1] &= 0 \end{aligned} \tag{5.19}$$

Next we construct a new parameterization of the surface (u, v) which is guaranteed to be smooth. Note that the map $\tilde{\Phi}(\xi) = (g^1, g^2)$ is homotopic to $\xi \to (\xi_1, \xi_2)$ and in sufficiently small neighborhood of zero is a covering projection everywhere except zero. Therefore, it is an injective mapping, as (ξ_1, ξ_2) is injective and also a covering projection. Thus, we can use $u = g_1(\xi), v = g_2(\xi)$ as our new coordinate functions in a neighborhood of zero. They are already known to be smooth everywhere except zero.

In the new coordinates the surface is simply a graph over the (u, v) plane.

From the conditions (5.19) it immediately follows that

$$\lim_{\|\xi\|_2 \to 0} \partial_u g^3(u, v) = 0$$
$$\lim_{\|\xi\|_2 \to 0} \partial_v g^3(u, v) = 0$$

Existence of derivatives equal to zero at 0 easily follows. We conclude that $g^3(u, v)$ is a smooth function.

□

Clearly, if all functions $f[x_{ij}](\Phi^{-1}(\xi))$ for $i > 2$ are C^k at 0 and have zero first partial derivatives, the surface is C^k continuous.

Necessity of this condition for C^k-continuity is also relatively straightforward. Any smooth 2-manifold immersed in $\mathbf{R}^3$ can be locally parameterized over its tangent plane, with parameterization being the inverse of the projection of the surface into the tangent plane. Consider a surface

$$F^{ij}(y) = (f_{ab}(y), f_{cd}(y), f[x_{ij}](y))$$

The tangent plane at 0 is the coordinate plane spanned by the first two coordinate vectors. The parameterization over the tangent plane is exactly $f[x_{ij}](\Phi^{-1}(\xi))$. If the surface is C^k, this function has to be C^k.

Thus, the problem of C^k continuity is reduced to analyzing C^k-continuity of functions $f[x_{ij}](\Phi^{-1}(\xi))$:

Theorem 5.2. *Let $\mathcal{S}$ be a stationary, affine invariant, locally defined subdivision scheme satisfying Conditions A and B on a k-regular complex. Let $\Phi(y)$ be the characteristic map of $\mathcal{S}$. Suppose that $\mathcal{S}$ is C^k on the regular complex. Then $\mathcal{S}$ is C^k on the k-regular complex if and only if the following conditions are satisfied:*
— *$\Phi(y_1) = \Phi(y_2)$, for all eigenbasis functions f_{ij} implies $f_{ij}(y_1) = f_{ij}(y_2)$.*
— *All functions $f_{ij}(\Phi^{-1}(\xi))$ are C^k continuous.*

Theorem 5.2 reduces the problem of checking C^k-continuity of a subdivision scheme to the problem of checking C^k-continuity of a finite number of special functions. More explicit conditions can be obtained if we further assume that the leading vectors x_{ab} and x_{cd} are actual eigenvectors, i.e. $b = d = 0$. It is possible to show that $b = 0$ actually has to be true for the continuous schemes, as we will see from the next theorem; d does not have to be 0. If $d \neq 0$, it is unclear if it is possible to formulate more explicit conditions for C^k-continuity.

Thus, for the following theorem we will assume

Condition C. Both vectors x_{ab} and x_{cd} are eigenvectors, i.e. $b = d = 0$.

In this case it follows from scaling relations of Theorem 4.1 that $f_{ij}(\xi) = f[x_{ij}](\Phi^{-1}(\xi))$ satisfy more general scaling relations of the form

$$f_{ij}(T\xi) = \lambda_i f_{ij}(\xi) + f_{i(j-1)}(\xi), \quad \text{for } j \geq 1$$
$$f_{i0}(T\xi) = \lambda_i f_{i0}(\xi)$$

where

$$T = \begin{pmatrix} \lambda_a & 0 \\ 0 & \lambda_b \end{pmatrix} \quad \text{or} \quad T = \begin{pmatrix} \lambda\cos\phi & -\lambda\sin\phi \\ \lambda\sin\phi & \lambda\cos\phi \end{pmatrix}$$

These relations are obtained simply by plugging in the relations of Theorem 4.1 for f_{a0} and f_{b0} into the relations for f_{ij}.

Functions satisfying scaling relations will be discussed in detail in the next section. Using the results obtained there, we can formulate a general criterion of C^k-continuity of subdivision schemes.

We won't state the most explicit form of the theorem, which is straightforward to obtain from our formulation and lemmas of the next section. In the case of a pair of real eigenvalues, rather then specifying conditions on the eigenbasis functions themselves, we will specify conditions on certain derivative of these functions.

Before stating the theorem, we need to define two special sets of polynomials which generalize the idea of homogeneous polynomials. These definitions differ only slightly from the standard definitions of *quasihomogeneous polynomials*, which are discussed in Section 6.3.

The set $\mathbf{N}(p,q)$ is the set of all pairs of integers (i,j) such that $\lambda_a^i\lambda_b^j = \lambda_a^p\lambda_b^q$ for a fixed pair (p,q). The set of polynomials $\mathbf{P}(p,q)$ is defined as

$$\mathbf{P}(p,q) = \left\{ \sum_{i,j} \alpha_{ij}\xi_1^i\xi_2^j \;\middle|\; (i,j) \in N(p,q),\ \alpha_{ij} \in \mathbf{C} \right\}$$

In the case of a pair of complex conjugate $\lambda_a = \boldsymbol{\lambda}$, $\lambda_b = \bar{\boldsymbol{\lambda}}$, a similar set $\bar{\mathbf{N}}(p,q)$ is the set of all pairs of integers (i,j) such that $\boldsymbol{\lambda}^i\bar{\boldsymbol{\lambda}}^j = \boldsymbol{\lambda}^p\bar{\boldsymbol{\lambda}}^q$ for a fixed pair (p,q). In this case we also define the set of polynomials

$$\bar{\mathbf{P}}(p,q) = \left\{ \sum_{i,j} \alpha_{ij}\xi_1^i\bar{\xi}_2^j \;\middle|\; (i,j) \in \bar{\mathbf{N}}(p,q),\ \alpha_{ij} \in \mathbf{C} \right\}$$

Finally, j_{min}^i for all λ_i satisfying $|\lambda_i| \geq |\lambda_b|^k$ is defined as

$$j_{min}^i = \min\left\{ j \;\middle|\; \exists l :\ l+j \leq k, \text{ and } |\lambda_a^l\lambda_b^j| \leq |\lambda_i| \right\}$$

An intuitive explanation of the meaning of this number will be given in the next section. Note that if $|\lambda_a| = |\lambda_b|$ and $|\lambda_i| \geq |\lambda_b^k|$, $j_{min}^i = 0$.

Theorem 5.3. *Suppose a subdivision scheme $\mathcal{S}$ converges on the k-regular complex and satisfies conditions A,B and C. Let Φ be the characteristic map of the scheme. Suppose the scheme is C^k-continuous on the regular complex. $\mathcal{S}$ is C^k-continuous on the k-regular complex if and only if the first condition of Theorem 5.2 is satisfied, and one of the following is true for any set of eigenbasis limit functions $f[x_{ij}](\Phi^{-1}(\xi)) = f_{ij}(\xi)$, $j = 1..n_j$ corresponding to the eigenvalue λ_i:*

1. *$f_{ij}(\xi) \equiv 0$,*
2. *OR $|\lambda_i| < |\lambda_b|^k$,*
3. *OR The leading eigenvalues λ_a and λ_b are real,*
   - *(a) $n_i = 0$, $\lambda_i = \lambda_a^p\lambda_b^{q+j_{min}}$ for some nonnegative p,q and $\partial_2^{j_{min}^i} f_{i0}(\xi) \in \mathbf{P}(p,q)$, $p+q \leq k - j_{min}^i$.*
   - *(b) OR $\partial_2^{j_{min}} f_{ij}(\xi) \equiv 0$ for all j.*
4. *OR the leading eigenvalues λ_a and λ_b are complex conjugate, $n_i = 0$, $\lambda_i = \lambda_a^p\lambda_b^q$ for some p,q, and*

$$f_{i0}(\xi) \in \bar{\mathbf{P}}(p,q), \quad p+q \leq k - j_{min}^i$$

The origin of the cases in Theorem 5.3 will become clear after Section 6. In the next section we provide a brief discussion of the result.

An important special case of Theorem 5.3 occurs when $\lambda_a = \lambda_b$; in this case the eigenvalues are necessarily real and the criterion becomes

Corollary 5.2. *If a subdivision scheme satisfies conditions of Theorem 5.3 and the leading eigenvalues of the subdivision matrix coincide, then the scheme is C^k if and only if any set of eigenbasis limit functions $f[x_{ij}]$ corresponding to an eigenvalue $\lambda_i \geq \lambda^k$ either consists of functions that are identically zero, or consists of one element f_{i0} such that $f_{i0}(\Phi^{-1}(\xi))$ is a homogeneous polynomial of degree d and $\lambda_i = \lambda^d$.*

5.5 Discussion

The complexity of the conditions of Theorem 5.3 is due to the fact that we consider non-equal, possibly complex, leading eigenvalues. This is a consequence of relaxing the definition of stationarity. If we assume that the scheme is stationary under any automorphisms of the k-regular complex, it is possible to show that the leading eigenvalues should have equal absolute values. Furthermore, for schemes with sufficiently small support the eigenvalues can be guaranteed to be real. These facts will be discussed in greater detail in a future paper. Thus, Corollary 5.2 is the most important case of the criterion. However, for schemes with large support the case of a pair of complex conjugate eigenvalues might occur, although it is not clear at this point if such schemes have any practical value.

The case of different real eigenvalues is important for subdivision schemes on tagged complexes, when only automorphisms preserving a subset of the complex are allowed. In this case the ratio of the eigenvalues can be used to obtain surfaces with a "smooth crease". The explicit expressions for eigenbasis functions in the criterion above provide a useful tool for constructing such schemes.

In addition to the case of equal real eigenvalues, the case of C^1 and C^2 continuity are of particular interest. We will state them in the case when the eigenvalues λ_a,λ_b are real.

C^1-continuity. If $|\lambda'| < |\lambda_b|$, existence of first derivatives is guaranteed. If $|\lambda'| \geq |\lambda_b|$, the conditions for C^1-continuity are

1. If $|\lambda_b| \leq |\lambda_i| < |\lambda_a|$, $\lambda_i \neq \lambda_b$, then for all j $f_{ij}(\xi) = f_j(\xi_1)$ and $f_j(\xi_1)$ satisfy the scaling relation for λ_i.
2. If $\lambda_i = \lambda_a$, $n = 0$, $f_{i0} = \alpha_{10}\xi_1$.
3. Otherwise $f_{ij} \equiv 0$ for all m.

These conditions on f_{ij} are in fact an explicit way to say that for all f_{ij} for which $|\lambda_i| \geq |\lambda_b|$, the Jacobian $J[f_{ij}, f_{ab}]$ should be zero. The case $\lambda_i = \lambda_b$ never occurs due to Condition B.

C^2-continuity. We will assume that $\lambda_a \neq \lambda_b$, Note that for any λ_a, λ_b, $|\lambda_b^2| < |\lambda_a\lambda_b| < |\lambda_a^2|$, $|\lambda_a\lambda_b| < |\lambda_b| < |\lambda_a|$. The relative position only two interesting powers of eigenvalues are not defined: λ_a^2 and λ_b. λ_a^2 can be either between $\lambda_a\lambda_b$ and λ_b, or between λ_b and λ_a.

1. If $\lambda_i = \lambda_b^2$, $n = 0$, then $f_{i0}(\xi) = \alpha_{00}(f_{00}^0(\xi_1) + f_{00}^1(\xi_1)\xi_2 + \xi_2^2)$, where f_{00}^0 satisfies the scaling relation for λ_b^2 and f_{00}^1 satisfies the scaling relation for λ_b.
2. If $|\lambda_b|^2 \leq |\lambda_i| < |\lambda_a\lambda_b|$ and $\lambda_i \neq \lambda_b^2$ or $n > 0$, then $f_{ij}(\xi) = f_j^0(\xi_1) + f_j^1(\xi_1)\xi_2$, where f_j^0 satisfy the scaling relation for λ_i and f_j^1 satisfy the scaling relation for $\frac{\lambda_i}{\lambda_b}$.
3. If $\lambda_i = \lambda_a\lambda_b$, $n = 0$, then $f_{i0}(\xi) = \alpha_{10}(f_{10}^0(\xi_1) + \xi_2)\xi_1 + f^0(\xi_1)$, where f_{10}^0 satisfies the scaling relation for λ_b and f^0 satisfies the scaling relation for $\lambda_a\lambda_b$.
4. If $|\lambda_a\lambda_b| \geq |\lambda_i| < |\lambda_1^2|$, $\lambda_i \neq \lambda_b$, or $n > 0$, then $f_{ij}(\xi) = f(\xi_1)$ and satisfy scaling relation for λ_i.
5. If $|\lambda_a\lambda_b| \geq |\lambda_i| < |\lambda_1^2|$, $\lambda_i = \lambda_b$, (this is possible only if $|\lambda_a^2| < |\lambda_b|$), and $n = 0$, then $f_{i0}(\xi) = \alpha_{00}(f_{00}^0(\xi_1) + \xi_2)$ where $f_{00}^0(\xi_1)$ satisfies scaling relation for λ_b.

6. If $\lambda_a = \lambda_b$, $n = 0$, there are two different cases to consider: $\lambda_a^2 = \lambda_b$ and $\lambda_a^2 \neq \lambda_b$. In the first case $f_{i0}(\xi) = \alpha_{20}\xi_1^2 + \alpha_{01}\xi_2$, in the second just $\alpha_{20}\xi_1^2$.

7. In all other cases $f_{ij}(\xi) \equiv 0$.

Note that as the eigenvalue λ_i increases from λ_b^2 to λ_a^2, the conditions on the eigenbasis function become more and more restrictive.

5.6 Degree Estimate for Piecewise polynomial Subdivision

A simple consequence of Theorem 5.3 is the following estimate for the minimal degree of the polynomial patches that is required to obtain C^k-smooth surfaces.

We derive these estimates for the case of two equal real eigenvalues, which is the most common case.

First, we note that if all eigenvalues are less then λ_b^k, then the scheme is C^k-continuous. But this also means that all partial derivatives up to the order k disappear at zero, i.e. the surface is "flat". Extending the definition of [17] we define a *non-flat surface* of order k as a C^k-continuous surface which has non-vanishing partial derivatives up to order k. In this case Corollary 5.2 tells us that the scheme should reproduce surfaces of the form

$$z = H^k(x, y)$$

where $H^k(x, y)$ is a homogeneous polynomial of degree k. x, y and z in the case of piecewise polynomial schemes, are piecewise polynomials of degrees less than $d > k$. As it was shown by Reif, the degree of piecewise polynomials x and y should be at least $k + 1$. Therefore, the degree of z is at least $k(k + 1)$.

6 Scaling relations

In this section we will derive several properties of *scaling relations*. The simplest form of a scaling relation in one dimension for one function $g(t)$ is

$$g(\lambda t) = \lambda' g(t)$$

The importance of relations of this type for subdivision was pointed out by Warren [24].

For our purposes, we need to generalize the relation above to the case of systems of functions defined on $\mathbf{R}^2$. We identify $\mathbf{C}$ and $\mathbf{R}^2$ and use boldface letters to denote the vectors in $\mathbf{R}^2$ and complex numbers in $\mathbf{C}$. ∂_1, ∂_2 denotes differentiation with respect to ξ_1 and ξ_2 respectively.

In a general form a scaling relation for a system of functions $g_m(\xi) : \mathbf{R}^2 \to \mathbf{C}$, $m = 0..n$, can be written as

$$\begin{aligned} g_m(T\xi) &= \lambda' g_m(\xi) + g_{m-1}(\xi), \quad \text{for } m \geq 1 \\ g_0(T\xi) &= \lambda' g_0(\xi) \end{aligned} \tag{6.1}$$

where T is a transformation of the plane. We will be interested in two special cases of (6.1): The case when T is a scaling (possibly non-uniform) in the directions ξ_1 and ξ_2, and the case when T is a uniform scaling combined with rotation:

$$T = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \quad \text{or} \quad T = \begin{pmatrix} \lambda\cos\phi & -\lambda\sin\phi \\ \lambda\sin\phi & \lambda\cos\phi \end{pmatrix} \tag{6.2}$$

The parameterizations generated by the leading eigenvectors corresponding to a pair of real eigenvalues satisfy the scaling relation of the first type, those corresponding to a pair of complex conjugate eigenvalues satisfy the scaling relation of the second type.

Particularly important special case of the scaling relation is the uniform scaling without rotation, which arises in the analysis of most known index invariant schemes.

In this section we prove criteria for C^k-continuity of functions satisfying scaling relations that were used in Section 5.

6.1 Nonuniform scaling

First we consider the scaling relation of the form

$$\begin{aligned} g_m(\lambda_1\xi_1, \lambda_2\xi_2) &= \lambda' g_m(\xi) + g_{m-1}(\xi), \quad \text{for } m \geq 1 \\ g_0(\lambda_1\xi_1, \lambda_2\xi_2) &= \lambda' g_0(\xi) \end{aligned} \tag{6.3}$$

We assume without the loss of generality that $|\lambda_1| \geq |\lambda_2|$. We say that a system of functions $g_m(\xi)$ satisfies (λ_1, λ_2)-scaling relation for λ' if it satisfies (6.3).

The following lemma is the basis of our derivations.

Lemma 6.1. *Suppose a system of functions $g_m(\xi)$, $m = 0..n$ satisfies the (λ_1, λ_2)-scaling relation for λ' and all functions g_m are defined on $D \setminus \{0\}$, where D is a compact domain in $\mathbf{R}^2$ which contains the origin as an internal point.*

1. *If $|\lambda'| < |\lambda_i|^k$, $i = 1, 2$, then*

$$\lim_{|\xi| \to 0} \frac{|g_m(\xi)|}{|\xi_i|^k} = 0$$

2. *If $\lambda' = 1$, and $n = 0$, then g_0 is continuous at 0 if and only if $g_0 = const$. If $n \neq 0$, then g_m are continuous if and only if all m $g_m \equiv 0$.*

3. If $|\lambda'| \geq 1$, and $\lambda' \neq 1$, then g_m are continuous if and only if $g_m \equiv 0$.

Proof. 1. Introduce a formal operator N, $Ng_j = g_{j-1}$ for $j > 0$, $Ng_0 = 0$. This operator commutes with multiplication by a constant and

$g_j(\lambda_1\xi_1, \lambda_2\xi_2) = (\lambda' I + N)g_j(\xi)$. Therefore

$$g_j(\lambda_1^p\xi_1, \lambda_2^p\xi_2) = \sum_{i=0}^{j} \binom{p}{i} (\lambda')^{p-i} g_{j-i}(\xi) \tag{6.4}$$

assuming $p \geq j$.

Suppose that the disk $|\xi| \leq r$ is contained in D.

Let

$$v_m = \max_{\substack{|\lambda_1| r \leq \xi_1 \leq r \\ |\lambda_2| r \leq \xi_2 \leq r}} g_m(\xi) \tag{6.5}$$

Then from (6.4) we have for $|\lambda_i^{p-1}| \leq |\xi_i| \leq |\lambda_i^p|$, $i = 1, 2$:

$$|g_j(\xi)| = \left| \sum_{i=0} \binom{p}{i} \lambda'^{p-j} p^j \right| \left| j \max_{0 \leq m \leq j} v_m \right| \leq \tilde{C} |\lambda'|^{p-j} p^j$$

where C and $\tilde{C}$ are constants. Therefore,

$$\left| \frac{g_j(\xi)}{|\xi_i|^k} \right| \leq C \left| \frac{\lambda'}{\lambda_i^k} \right|^p \frac{p^j}{|\lambda'|^j}$$

The right-hand side goes to 0 as $p \to \infty$. Thus,

$$\lim_{|\xi| \to 0} \frac{|g_m(\xi)|}{|\xi_i|^k} = 0$$

2. Assume that $g_0(\xi)$ is not constant. Suppose $g_0(\xi) - g_0(\xi') = d \neq 0$. Then $g_0(\lambda_1^k\xi_1, \lambda_2^k\xi_2) - g_0(\lambda_1^k\xi_1', \lambda_2^k\xi_2') = d \neq 0$. Clearly, the limit of $g(\xi)$ does not exist at 0. If $|\lambda'| = 1$ but $\lambda' \neq 1$, if $g_0(\xi) \neq 0$, $g_0(\lambda_1\xi_1, \lambda_2\xi_2) = \lambda' g_0(\xi) \neq g_0(\xi)$. Hence, in this case $g_0(\xi)$ should be identically zero.

   For $m \geq 0$, from $g_m(\lambda_1\xi_1, \lambda_2\xi_2) = g_m(\xi) + g_{m-1}(\xi)$, it follows that $g_j(0) = 0$ for $j < m$. Therefore, $\lim_{|\xi| \to} g_0(\xi) = 0$ for all ξ, if it is continuous. Then $g_1(\xi)$ satisfies $g_1(\lambda_1\xi_1, \lambda_2\xi_2) = g_1(\xi)$ and part 2 applies again. By induction, all $g_m(\xi)$ are identically zero. The converse is trivial.

3. If $g_0(\xi) \not\equiv 0$, $|g_0(\lambda_1^k\xi_1, \lambda_2^k\xi_2)| \to \infty$ as $k \to \infty$, hence the limit does not exist.

   If $\lim_{|\xi| \to 0} g_m(\xi)$ exists, $\lim_{|\xi| \to 0} g_{m-1}(\xi) = \lim_{|\xi| \to 0} g_m(\lambda_1\xi_1, \lambda_2\xi_2) - \lambda' g_m(\xi)$ exists. By induction, we conclude that $\lim_{|\xi| \to 0} g_0(\xi)$ has limit at 0. Therefore, it is identically 0. Then similar to the previous case, we can show that all functions g_j, $j \leq m$ should be 0.

□

The derivatives of functions satisfying a scaling relation don't satisfy a scaling relation themselves, but their scaled versions do. This allows us to establish the following

Lemma 6.2. *If a system of functions satisfies the (λ_1, λ_2)-scaling relation for λ', the functions are defined on a compact domain containing 0 as an internal point, and are C^k everywhere except 0, then the derivative $\partial_1^i \partial_2^j g_m(\xi)$, $i + j \leq k$, exists at 0 and is continuous if and only if one of the following conditions is met:*

1. $|\lambda'| < |\lambda_1^i \lambda_2^j|$

2. $\lambda' = \lambda_1^i \lambda_2^j$, $n = 0$, *and* $\partial_1^i \partial_2^j g_0(\xi) \equiv const$, *or* $n > 0$ *and* $\partial_1^i \partial_2^j g_m(\xi) \equiv 0$ *for all* m.

3. $|\lambda'| \geq \left|\lambda_1^i \lambda_2^j\right|$, $\lambda' \neq \lambda_1^i \lambda_2^j$ *and* $\partial_1^i \partial_2^j g_m(\xi) \equiv 0$ *for all* m.

Proof. Differentiating the scaling relation, we obtain

$$\lambda_1^i \lambda_2^j \partial_1^i \partial_2^j g_m(\lambda_1 \xi_1, \lambda_2 \xi_2) = \lambda' \partial_1^i \partial_2^j g_m(\xi) + \partial_1^i \partial_2^j g_{m-1}(\xi) \quad \text{for } m > 0$$
$$\lambda_1^i \lambda_2^j \partial_1^i \partial_2^j g_0(\lambda_1 \xi_1, \lambda_2 \xi_2) = \lambda' \partial_1^i \partial_2^j g_0(\xi)$$

Define the functions

$$\tilde{g}_m(\xi) = \frac{1}{(\lambda_1^i \lambda_2^j)^m} \partial_1^i \partial_2^j g_m(\xi)$$

Then $\tilde{g}_m(\xi)$ satisfy

$$\tilde{g}_m(\lambda_1 \xi_1, \lambda_2 \xi_2) = \frac{\lambda'}{\lambda_1^i \lambda_2^j} \tilde{g}_m(\xi) + \tilde{g}_{m-1}(\xi) \quad \text{for } m > 0$$
$$\tilde{g}_0(\lambda_1 \xi_1, \lambda_2 \xi_2) = \tilde{g}_0(\xi)$$

which is the (λ_1, λ_2)-scaling relation for $\frac{\lambda'}{\lambda_1^i \lambda_2^j}$.

As the functions $\tilde{g}_m$ are just scaled versions of $\partial_1^i \partial_2^j g_m$, their properties are the same.

Now we can apply Lemma 6.1.

Necessity immediately follows from Lemma 6.1, as the derivatives $\partial_1^i \partial_2^j g_m(\xi)$ and, therefore, $\tilde{g}_m(\xi)$ have to be continuous.

Sufficiency.

1. If both i and j are equal to zero, the statement of the lemma reduces to the statement of Lemma 6.1.
   Assuming that $j \neq 0$, consider the derivative $\partial_1^i \partial_2^{j-1} g_m$. If $\left|\frac{\lambda'}{\lambda_1^i \lambda_2^j}\right| < 1$, then $\left|\frac{\lambda'}{\lambda_1^i \lambda_2^{j-1}}\right| < |\lambda_2|$. As the set of functions $\tilde{g}_m$ corresponding to $\partial_1^i \partial_2^{j-1} g_m$ satisfies the (λ_1, λ_2)-scaling relation for $\frac{\lambda'}{\lambda_1^i \lambda_2^{j-1}}$, by part 1 of Lemma 6.1, the limit

   $$\lim_{|\xi| \to 0} \frac{g_m(\xi)}{\xi_2}$$

   exists and is equal to 0. This limit is precisely the derivative $\partial_1^i \partial_2^j g_m(\xi)$.

   Similarly, as $\left|\frac{\lambda'}{\lambda_1^i \lambda_2^j}\right| < 1$, the limit of $\partial_1^i \partial_2^j g_m(\xi)$ exists and is equal to 0. We conclude that the derivative $\partial_1^i \partial_2^j g_m(\xi)$ exists and is continuous.

2. If $\frac{\lambda'}{\lambda_1^i \lambda_2^j} = 1$, and $n = 0$, to be continuous, $\partial_1^i \partial_2^j g_m$ has to be constant. Therefore, $\partial_1^i \partial_2^{j-1} g_m$ is linear in ξ_2, for $\xi \neq 0$, and the appropriate limit clearly exists, and is equal to the same constant. Thus, the derivative in this case exists. If $n = 0$, $\partial_1^i \partial_2^j g_m$ is 0 everywhere. Then $\partial_1^i \partial_2^{j-1} g_m$ is a constant for $\xi \neq 0$, and $\partial_1^i \partial_2^j g_m$ exists and is 0 at $\xi = 0$. This argument also applies to part 3.

$\square$

The functions $g_m(\xi)$ are C^k-continuous if all derivatives

$$\partial_1^i \partial_2^j g_m(\xi)$$

with $i+j \leq k$ exist and are continuous. The derivative $\partial_1^i \partial_2^j g_m$ can be associated with the integer point (i,j) in the plane. Such representation is used for *Newton diagrams* of quasihomogeneous polynomials (Section 6.3.)

We are interested in the existence and continuity of the derivatives which are represented by integer points inside the triangle bounded by $x=0$, $y=0$, $x+y=k$ (Figure 6.1).

According to Lemma 6.2, the derivatives $\partial_1^i \partial_2^j g_m$ are guaranteed to exist at 0 if $\left|\frac{\lambda'}{\lambda_1^i \lambda_2^j}\right| < 1$. Taking logarithms of both sides of this inequality, we can see that for all integer points below the line $l(\lambda')$ with equation $x \ln|\lambda_1| + y \ln|\lambda_2| = \ln|\lambda'|$, the derivatives are known to exist. The slope of $l(\lambda')$ is less then -1, because $|\lambda_2| < |\lambda_1|$.

For the points between the lines $l(\lambda')$ and $x+y=k$, the derivatives have to be either 0 or constants to exist and be continuous. For those that are constants additional conditions $\lambda' = \lambda_1^i \lambda_2^j$ and $n=0$ have to be satisfied.

Note that if a derivative $\partial_1^i \partial_2^j g_m$ is 0 or constant, all derivatives to the right and upward from (i,j) are equal to zero everywhere. Suppose $|\lambda'| \geq |\lambda^k|$; this means that $l(\lambda')$ intersects the y axis below or at the point $(0,k)$.

In this case let j_{min} be the minimal integer value of y for which there is an integer point (x, j_{min}) between $l(C)$ and $x+y=k$. All derivatives represented by integer points inside the area delimited by $x=0$, $l(\lambda')$, $x+y=k$, $y=j_{min}$ are 0 (shaded area in the Figure 6.1).

Before formulating the result following from these considerations, we need to introduce the sets of polynomials $\mathbf{P}(p,q)$, generalizing the idea of homogeneous polynomials to the case of nonuniform scaling of variables. Let $\mathbf{N}(p,q)$ be the set of all pairs of integers (i,j) such that $\lambda_1^i \lambda_2^j = \lambda_1^p \lambda_2^q$ for a fixed pair (p,q) Then

$$\mathbf{P}(p,q) = \left\{ \sum_{i,j} \alpha_{ij} \xi_1^i \xi_2^j \;\middle|\; (i,j) \in N(p,q),\ \alpha_{ij} \in \mathbf{C} \right\}$$

Lemma 6.3. *Suppose a set of functions $g_m(\xi)$, $m = 0 \ldots n$ satisfies the conditions of Lemma 6.2. If $|\lambda'| \geq |\lambda_2|^k$, set $j_{min} = \min\{j \in \mathbf{N} \mid \left|\lambda_1^i \lambda_2^j\right| \leq |\lambda'|,\ i+j \leq k, \quad \text{for some } i \in N\}$.*

All functions are C^k-continuous at 0 if and only if one of the following conditions holds

1. $|\lambda'| < |\lambda_2|^k$,
2. $n = 0$ *and* $\lambda' = \lambda_1^p \lambda_2^{q+j_{min}}$ *for some* p,q *and* $\partial_2^{j_{min}} g_0(\xi) \in \mathbf{P}(p,q)$, $p+q \leq k - j_{min}$.
3. $\partial_2^{j_{min}} g_m(\xi) \equiv 0$ *for all* m.

Proof.
Necessity. Suppose $|\lambda'| \geq |\lambda_2|^j$. As it was observed above, all derivatives corresponding to the integer points in the area between $l(\lambda')$ and $x+y=k$ should be constant. Note that all derivatives $\partial_1^i \partial_2^{j+j_{min}} g_0$ for all m and for $i+j > k - j_{min}$ are 0. This means that $\partial_2^j g_0$ is a polynomial because all its derivatives of order $k - j_{min}$ are zeros.

Suppose $\partial_1^p \partial_2^{q+j_{min}} g_m$ is not zero for some p and q satisfying $p+q \leq k - j_{min}$. Then it follows from Lemma 6.2 that $\lambda' = \lambda_1^p \lambda_2^q$ and $n=0$.

Moreover, as this polynomial should satisfy the scaling relation for λ' which is equal to $\lambda_1^p \lambda_2^q$, it is easy to see that it should be contained in $P(p,q)$.

If all derivatives including $\partial_2^{j_{min}} g_m$ are zeros, the third condition is satisfied.

Sufficiency. If the first condition of the lemma is satisfied, the derivatives up to order k exist by Lemma 6.2.

Suppose the second or the third condition is satisfied. Then all derivatives $\partial_1^i \partial_2^j g_m$ exist for $j \geq j_{min}$. However, if $j < j_{min}$, the scaled versions of the derivatives $\tilde{g}_m$ satisfy scaling relation for $\frac{\lambda'}{\lambda_1^i \lambda_2^j}$, and $\left|\frac{\lambda'}{\lambda_1^i \lambda_2^j}\right| < 1$ by definition of j_{min}. Therefore, all other derivatives also exist.

□

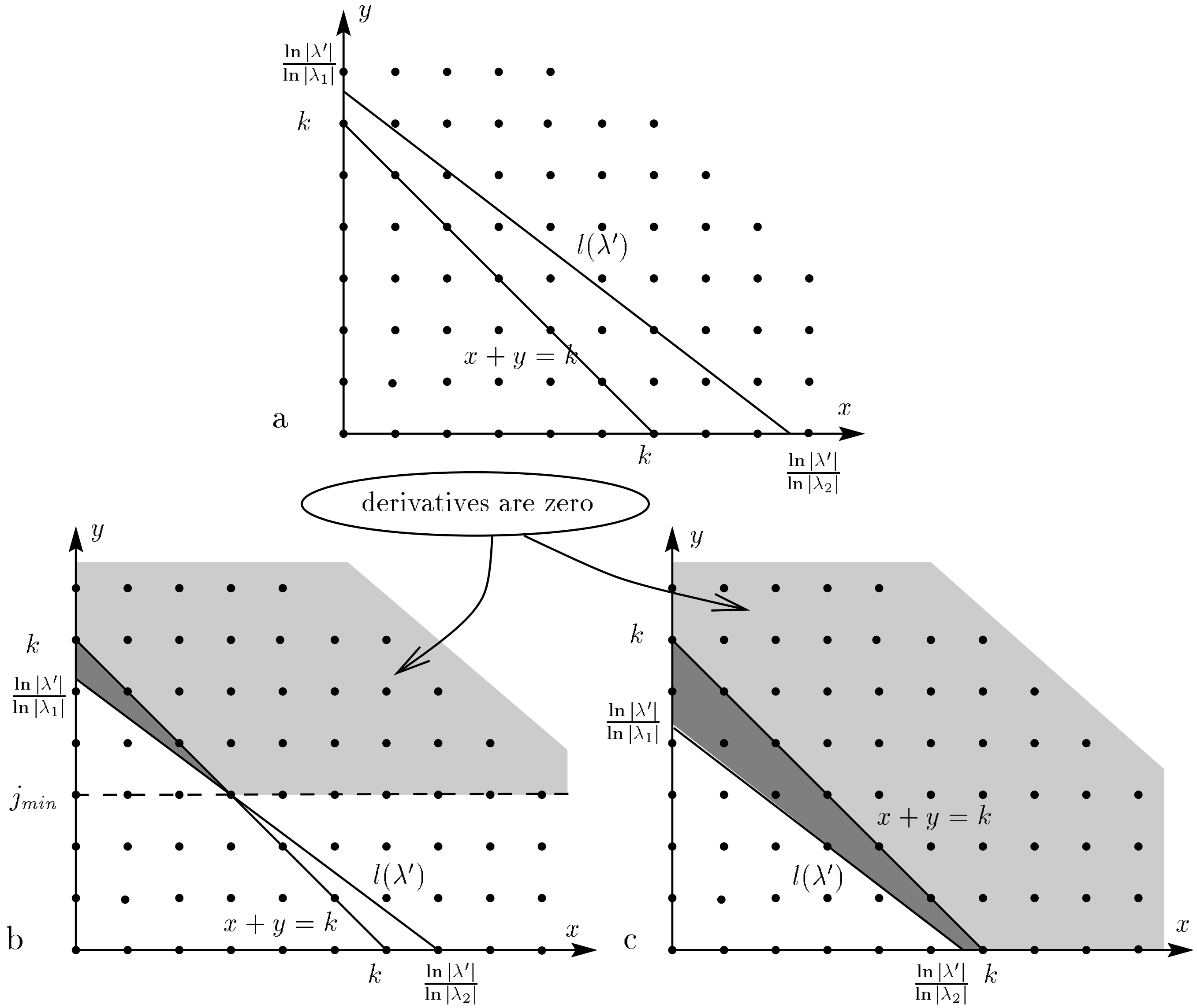

Figure 22: (a) $|\lambda'| < |\lambda_2|^k$; All variables up to order k exist. (b) $|\lambda'| > |\lambda_2|^k$, $j_{min} = 0$; The function g_m has to be a polynomial. (c) $|\lambda'| > |\lambda_2|^k$, $j_{min} \neq 0$; derivative $\partial_2^{j_{min}}$ has to be a polynomial.

The condition on $\partial_2^{j_{min}} g_m$ does not give the explicit form for the functions g_m unless $j_{min} = 0$. It is possible to find a more explicit expression for g_m that are C^k and satisify scaling relation for λ'.

Lemma 6.4. *A set of functions $g_m(\xi)$, $m = 0 \dots n$ that are C^k-continuous on a set $D \setminus \{0\}$, where D is a compact set containing zero as an internal point satisfies the (λ_1, λ_2)-scaling relation and is C^k-continuous at zero, if and only if one of the following conditions is satisfied:*

1. $|\lambda'| < |\lambda_2|^k$,
2. $n = 0$, $\lambda' = \lambda_1^p \lambda_2^{q+j_{min}}$ *for some nonnegative* p, q *and if* $i \neq 0$ *for all* $(i,j) \in \mathbf{N}(p,q)$ *then*

$$g_0(\xi) = \sum_{i,j \in \mathbf{N}(p,q)} \alpha_{ij} \xi_1^i \left(\sum_{s=0}^{j_{min}-1} f_{ij}^s(\xi_1) \xi_2^s + \frac{\xi_2^{j+j_{min}}}{(j + j_{min})!} \right) + \sum_{s=0}^{j_{min}-1} f^s(\xi_1) \xi_2^s$$

if $(0, j) \in \mathbf{N}(p,q)$ *for some* j *then*

$$g_0(\xi) = \sum_{i,j \in \mathbf{N}(p,q)} \alpha_{ij} \xi_1^i \left(\sum_{s=0}^{j_{min}-1} f_{ij}^s(\xi_1) \xi_2^s + \frac{\xi_2^{j+j_{min}}}{(j + j_{min})!} \right)$$

where f_{ij}^s *are* C^k *and satisfy the scaling relation for* $\lambda_2^{j+j_{min}-s}$, f^s *satisfy the scaling relation for* $\lambda^p \lambda^{q+j_{min}-s}$

3.
$$g_m(\xi) = \sum_{s=0}^{j_{min}-1} f_m^s(\xi_1) \xi_2^s$$

for $m = 0..n$, *where* f_m^s *are* C^k *and their scaled variants* $\tilde{f}_m^s = \lambda_2^{-sm} f_m^s$ *satisfy the scaling relation for* $\frac{\lambda'}{\lambda_2^s}$

Proof.

Sufficiency is proved by checking that the functions above satisfy the scaling relation.

We will prove necessity for the third condition. The proof for the second condition is similar, although the expressions are more complicated.

Suppose $n > 0$. In this case by Lemma 6.3, $\partial_2^{j_{min}} g_m \equiv 0$ for all m.

Integrating with respect to ξ_2 the expression $\partial_2^{j_{min}}$, we get the expression for g_m as in the third condition of the lemma.

Fixing xi_2, at a set of values $x_1 \dots x_{j_{min}}$ such that the matrix

$$\begin{pmatrix} 1 & x_1 & x_1^2 & \dots & x_1^{j_{min}-1} \\ 1 & x_2 & x_2^2 & \dots & x_2^{j_{min}-1} \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ 1 & x_{j_{min}} & x_{j_{min}}^2 & \dots & x_{j_{min}}^{j_{min}-1} \end{pmatrix}$$

has nonzero determinant, we can express $f_m^s(\xi_1)$ as linear combinations of $g_m(\xi_1, \xi_2)$ with $\xi_2 \in \{x_1 \dots x_{j_{min}}\}$. Therefore, the functions have to be C^k if g_m is C^k.

The expression for $\partial_2^j g_m$ is

$$\partial_2^j g_m(\xi) = \sum_{i=j}^{j_{min}-1} f_m^s(\xi_1) \frac{i!}{(i-j)!} \xi_2^{i-j} \tag{6.6}$$

For $j = j_{min} - 1$, we get $\partial_2^{j_{min}-1} = (j_{min}-1)! f_m^{j_{min}-1}(\xi_1)$. Therefore, the set of functions $f_m^{j_{min}-1}(\xi_1)$ should satisfy the scaling relation for $\frac{\lambda'}{\lambda_2^{j_{min}-1}}$ by Lemma 6.2.

Suppose that $\tilde{f}_m^s$ satisfy the scaling relations for $\frac{\lambda'}{\lambda_2^s}$ for $s < j$. As g_m satisfies the scaling relation for λ', its derivative should satisfy scaling relation for $\frac{\lambda'}{\lambda_2^j}$. Using (6.6), we can rewrite the scaling relation as

$$\sum_{i=j}^{j_{min}} f_m^s(\lambda_1\xi_1)\xi_2^{i-j}\lambda_2^{i-j}\frac{i!}{(i-j)!} = \frac{\lambda'}{\lambda_2^j}\sum_{i=j}^{j_{min}} f_m^s(\xi_1)\xi_2^{i-j}\frac{i!}{(i-j)!} + \frac{1}{\lambda_2^j}\sum_{i=j}^{j_{min}} f_{m-1}^s(\xi_1)\xi_2^{i-j}\frac{i!}{(i-j)!} \qquad (6.7)$$

Regrouping and substituting expressions for $\tilde{f}$ results in

$$\sum_{i=j}^{j_{min}} \lambda_2^{i-j}\frac{i!}{(i-j)!}\left(\tilde{f}_m^s(\lambda_1\xi_1)\xi_2^{i-j} - \frac{\lambda'}{\lambda_2^i}\tilde{f}_m^s(\xi_1)\xi_2^{i-j} + \tilde{f}_{m-1}^s(\xi_1)\xi_2^{i-j}\right) = 0 \qquad (6.8)$$

In the last equation all terms in the sum are 0 by assumption, with the exception of the term $i = j$. For the sum to be zero this term has to be zero, which means that $\tilde{f}_m^j$ satisfy the scaling relation for $\frac{\lambda'}{\lambda_2^j}$.

This proves that in the case $n > 0$ the third condition is necessary.

□

If $\lambda_1 = \lambda_2$, it is guaranteed that $j_{min} = 0$. Moreover, the set $\mathbf{P}(p,q)$ becomes simply the set of homogeneous polynomials of a given degree $p+q$.

This case is particularly important for the analysis of subdivision schemes, so we state is as a separate corollary.

Corollary 6.1. *Suppose a set of functions $g_m(\xi)$, $m = 0\ldots n$ satisfies the conditions of Lemma 6.2 for $\lambda_1 = \lambda_2 = \lambda$. All functions are C^k-continuous at 0 if and only if one of the following conditions holds*

1. $|\lambda'| < |\lambda|^k$,
2. $n = 0$ *and* $\lambda' = \lambda^p$ *for some* p *and* $g_0(\xi)$ *is a homogeneous polynomial of degree* j.
3. $g_m(\xi) \equiv 0$ *for all* m.

6.2 Uniform scaling and rotation

In this section we will consider the other case in (6.2), namely, the case of scaling relation

$$\begin{aligned} g_m(T\xi) &= \lambda' g_m(\xi) + g_{m-1}(\xi), \quad \text{for} m \geq 1 \\ g_0(T\xi) &= \lambda' g_0(\xi) \end{aligned}$$

with

$$T = \begin{pmatrix} \lambda\cos\phi & -\lambda\sin\phi \\ \lambda\sin\phi & \lambda\cos\phi \end{pmatrix}$$

In this case it is convenient to consider ξ to be a complex number.

We introduce operators $\partial = \partial_1 - i\partial_2$ and $\bar{\partial} = \partial_1 + i\partial_2$. Clearly, any derivative $\partial_1^i\partial_2^j$ can be expressed as a linear combination of operators $\partial^s\bar{\partial}^t$ with $s + t = i + j$. We will examine conditions for existence and continuity of derivatives of the form $\partial^i \bar{\partial}^j$, for $i + j \leq k$, which is equivalent to C^k continuity.

Observe that $T\xi = \boldsymbol{\lambda}\xi$, where $\boldsymbol{\lambda} = \lambda \exp i\phi$.

Thus, the scaling relation takes the form

$$\begin{aligned} g_m(\boldsymbol{\lambda}\xi) &= \lambda' g_m(\xi) + g_{m-1}(\xi), \quad \text{for} m \geq 1 \\ g_0(\boldsymbol{\lambda}\xi) &= \lambda' g_0(\xi) \end{aligned} \tag{6.9}$$

If the system of functions $g_m(\xi)$ satisfies (6.9) we will say that it satisfies $\boldsymbol{\lambda}$-scaling relation for λ'.

The following two equations make the transfer of Lemmas 6.1-6.4 a straightforward operation:

$$\begin{aligned} \partial(f(\boldsymbol{\lambda}\xi)) &= \boldsymbol{\lambda}(\partial f)(\boldsymbol{\lambda}\xi) \\ \partial(f(\boldsymbol{\lambda}\xi)) &= \bar{\boldsymbol{\lambda}}(\partial f)(\boldsymbol{\lambda}\xi) \end{aligned} \tag{6.10}$$

Lemma 6.1 becomes

Lemma 6.5. *Suppose a system of functions $g_m(\xi)$, $m = 0..n$ satisfies the $\boldsymbol{\lambda}$-scaling relation for λ' and all functions g_m are defined on $D \setminus \{0\}$, where D is a compact domain in $\mathbf{R}^2$ which contains the origin.*

1. *If $|\lambda'| < |\boldsymbol{\lambda}|^k$, then*

$$\lim_{|\xi| \to 0} \frac{|g_m(\xi)|}{|\xi_i|^k} = 0$$

for $i = 1, 2$.

2. *If $\lambda' = 1$, and $n = 0$, then g_0 is continuous at 0 if and only if $g_0 = const$. If $n \neq 0$, then g_m are continuous if and only if all m $g_m \equiv 0$.*

3. *If $|\lambda'| \geq 1$, and $\lambda' \neq 1$, then g_m are continuous if and only if $g_m \equiv 0$.*

The proof of this lemma is practically identical to the proof of Lemma 6.1; Instead of $|\lambda_1|$ and $|\lambda_2|$ we use $|\boldsymbol{\lambda}|$. For example, the definition of v (6.5) becomes

$$v_m = \max_{\substack{|\boldsymbol{\lambda}| r \leq \xi_1 \leq r \\ |\boldsymbol{\lambda}| r \leq \xi_2 \leq r}} g_m(\xi)$$

The rest of the proof is changed in a similar way.

Lemma 6.2 becomes

Lemma 6.6. *If a system of functions satisfies the $\boldsymbol{\lambda}$-scaling relation for λ', the functions are defined on a compact domain containing 0 as an internal point, and are C^k everywhere except 0, then the derivative $\partial^i \bar{\partial}^j g_m(\xi)$, $i + j \leq k$, exists at 0 and is continuous if and only if one of the following conditions is met:*

1. *$|\lambda'| < |\boldsymbol{\lambda}|^{i+j}$*

2. *$\lambda' = \boldsymbol{\lambda}^i \bar{\boldsymbol{\lambda}}^j$, $n = 0$, and $\partial_1^i \partial_2^j g_0(\xi) \equiv const$, or $n > 0$ and $\partial^i \bar{\partial}^j g_m(\xi) \equiv 0$ for all m.*

3. *$|\lambda'| \geq |\boldsymbol{\lambda}|^{i+j}$, $\lambda' \neq \boldsymbol{\lambda}^i \bar{\boldsymbol{\lambda}}^j$ and $\partial_1^i \partial_2^j g_m(\xi) \equiv 0$ for all m.*

The proof of this lemma exactly repeats the proof of Lemma 6.2, if λ_1 and λ_2 are replaced with $\boldsymbol{\lambda}$ and $\bar{\boldsymbol{\lambda}}$ respectively, and ∂_1 and ∂_2 are replaced with ∂ and $\bar{\partial}$.

The analog of Lemma 6.3 is much simpler due to the fact that $|\boldsymbol{\lambda}| = |\bar{\boldsymbol{\lambda}}|$, and is more similar to the Corollary 6.1. There is no need for an analog of Lemma 6.4, as Lemma 6.7 gives direct conditions

We define $\bar{\mathbf{N}}(p,q)$ and $\bar{\mathbf{P}}(p,q)$ similar to $\mathbf{N}(p,q)$ and $\mathbf{P}(p,q)$. Let $\bar{\mathbf{N}}(p,q)$ be the set of all pairs of integers (i,j) such that $\boldsymbol{\lambda}^i\bar{\boldsymbol{\lambda}}^j = \boldsymbol{\lambda}^p\bar{\boldsymbol{\lambda}}^q$ for a fixed pair (p,q) Then

$$\bar{\mathbf{P}}(p,q) = \left\{ \sum_{i,j} \alpha_{ij}\xi_1^i\bar{\xi}_2^j \;\middle|\; (i,j) \in \bar{\mathbf{N}}(p,q),\ \alpha_i j \in \mathbf{C} \right\}$$

Lemma 6.7. *Suppose a set of functions $g_m(\xi)$, $m = 0 \ldots n$ satisfies the conditions of Lemma 6.2. All functions are C^k-continuous at 0 if and only if one of the following conditions holds*

1. $|\lambda| < |\boldsymbol{\lambda}|^k$,
2. $n = 0$ *and* $\lambda' = \boldsymbol{\lambda}^p\bar{\boldsymbol{\lambda}}^q$ *for some* p, q *and* $g_0(\xi) \in \bar{\mathbf{P}}(p,q)$, $p + q \le k$.
3. $g_m(\xi) \equiv 0$ *for all* m.

The crucial observation is that in this case j_{min} is always 0. Using the same substitutions as for Lemma 6.6, we obtain the proof of the last lemma from the proof of Lemma 6.3.

6.3 Classification of Quasihomogeneous Polynomials

The conditions provided by Lemmas 6.3 and 6.7 are not explicit in one aspect: they use implicitly defined sets of polynomials $\mathbf{P}(p,q)$. If a set $\mathbf{P}(p,q)$ consists of more then one polynomial, it is possible to enumerate all polynomials in this set as points on a line with a rational slope in the Newton plane. It would be of some interest however, to find classes of polynomials that can be reduced to each other with suitable coordinate transformations.

Polynomials in these sets play an important role in singularity theory. In this section we will briefly touch on the subject; the nature and origin of the relation between quasihomogeneous functions and eigenbasis functions of subdivision is not clear and requires further study.

First we make the connection between functions satisfying scaling relations and quasihomogeneous functions explicit.

Definition 6.1. *A function satisfying relation*

$$f(\lambda^{q_1}\xi_1, \lambda^{q_2}\xi_2) = \lambda f(\xi_1, \xi 2) \tag{6.11}$$

where $q_1 + q_2 = 1$, is called quasihomogeneous function of degree 1 with exponents q_1, q_2.

It is clear that whenever the set $P(p,q)$ is not spanned by one monomial, there is a non-zero pair of integers p_1, p_2 such that $\lambda_1^{p_1}\lambda_2^{p2} = 1$. Therefore, all scaling factors in the corresponding relation can be expressed in terms of $\lambda_1^p\lambda_2^q$, resulting in the relation of the type (6.11). The exponents q_1 and q_2 in this case will be positive and rational, which means that any smooth quasihomogeneous function satisfying (6.11) has to be a polynomial.

Thus, the set of quasihomogeneous functions with given positive rational exponents is exactly one of the sets $P(p,q)$.

With each monomial of a quasihomogeneous polynomial we can associate a pair of integer points in the plane, similar to the way it was done in Figure 6.1.

All such points (i,j) are on the line $q_1 i + q_2 j = 1$. Monomials corresponding to these points are called diagonal. Monomials below the line are called lower, monomials above the line are called upper.

An important quantity characterizing quasihomogeneous functions is *intrinsic modality.* To define intrinsic modality, we first define the local algebra of a quasihomogeneous function f. This algebra is the factor algebra of all formal power series with respect to the ideal generated by the partial derivatives of f.

The local algebra has a basis of monomials, and the number of upper, lower and diagonal monomials in the basis does not depend on the choice of the basis.

The *intrinsic* or *inner* modality is the number of upper and diagonal monomials of any basis of the local algebra. There is a simple geometric method for calculating modality on the exponent plane.

Any quasihomogeneous polynomial of given type can be represented in the form $f_0 + \sum c_k e_k$, where f_0 is a fixed quasihomogeneous polynomial of the same type, e_k are diagonal monomials of the basis of the local algebra and c_k are some coefficients.

The sets $\mathbf{P}(p, q)$ can be specified by presenting the diagonal monomials of the basis of the local algebra and one quasihomogeneous polynomial with given exponents.

The equivalence of quasihomogeneous functions is given by the group of quasihomogeneous diffeomorphisms. The classification of the quasihomogeneous functions of intrinsic modality 0 up to equivalence is particularly simple: all such functions are listed in the table below (up to renaming of variables):

Type	Normal Form	q_1	q_2
A_k	$ax^{k+1} + by^1$	$k+1$	$\frac{1}{2}$
D_k	$ax^2y + by^{k-1}$	$\frac{k-2}{2k-2}$	$\frac{2}{2k-2}$
E_6	$ax^3 + by^4$	$\frac{1}{3}$	$\frac{1}{4}$
E_7	$ax^3 + bxy^3$	$\frac{1}{3}$	$\frac{2}{9}$
E_8	$ax^3 + by^5$	$\frac{1}{3}$	$\frac{1}{5}$

The same 5 classes occur in various classifications, such as classifications of simple Lie algebras, braid groups, regular polyhedra.

Many references to the singularity theory literature can be found in Arnold [2].

7 Injectivity of the Characteristic Map

In Section 5, the question of C^1 regularity of a subdivision scheme was reduced to checking local regularity of the characteristic map and compatibility of eigenbasis functions. From the practical point of view, injective characteristic maps are most useful. If the characteristic map is injective with non-zero Jacobian everywhere it is defined, it is sufficient for smoothness.

We show that in general injectivity of the characteristic map can be inferred from injectivity of a sufficiently close linear approximation.

7.1 Lipschitz Norms

In this section we state several simple facts about Lipschitz continuous functions with a special structure and prove a useful injectivity criterion.

Lemma 7.1. *For a C^1 function f defined on a compact convex domain,*

$$||f||_{Lip} = ||||\nabla f||_2||_\infty$$

Proof. $||f(x_1) - f(x_2)||_2 = ||f'_\tau(\hat{x})||_2$ where f'_τ is the derivative in the direction $\tau = \frac{x_2 - x_1}{||x_2 - x_1||_2}$ $\hat{x}$ is a point of the line segment $[x_1, x_2]$.

$||f'_\tau(\hat{x})||_2 \le ||\nabla f||_2$, and equality is achieved for a direction τ_0.

On the other hand, for $x_2 = x_1 + \tau_0 t$, $\lim_{t\to 0} \frac{||f(x_1) - f(x_2)||_2}{||x_1 - x_2||_2} = ||\nabla f||_2$. Hence, $||f||_{Lip} = ||||\nabla f||_2||_\infty$.

□

Lemma 7.2. *If a function is piecewise C^1 on a convex union A of convex domains, and on each domain it is C^1, with one-sided limits of the gradient existing on the boundary, and for any x_1, x_2 the line segment $[x_1, x_2]$ intersects only a finite number of domain boundaries, then*

$$||f||_{Lip} < ||||\nabla f||_2||_\infty$$

Proof. Let $b_0 = x_1, b_1, \ldots b_n = x_2$ be intersections of $[x_1, x_2]$ with domain boundaries. Then

$$||f(x_1) - f(x_2)||_2 \le \sum_{i=0}^{n-1} ||f(b_{i+1}) - f(b_i)||_2 \le ||||\nabla f||_2||_\infty \sum_i ||b_{i+1} - b_i||_2 = ||||\nabla f||_2||_\infty ||x_1 - x_2||_2$$

.

□

Lemma 7.3. *Assume that a function is piecewise C^1 on a union of convex domains and on each domain it is C^1, with one-sided limits of the gradient existing on the boundary. In addition, assume that there is a constant B, such that for any x_1 x_2 there is a piecewise linear path in A connecting x_1 and x_2 of length no more then $B||x_1 - x_2||_2$. Then*

$$||f||_{Lip} < B||||\nabla f||_2||_\infty$$

The proof is similar to the proof of Lemma 7.2.

Lemma 7.4. *Let Ω be a compact set. Suppose g is Lipschitz continuous on Ω. If L^j is a sequence of functions converging to g in the Lipschitz norm, and $0 < K' \le \left\|L^j\right\|_{Lip} \le K$ for some K. Then g is injective.*

Proof. As for any x_1, x_2

$$0 < K' \le \frac{\left\|L^j(x_1) - L^j(x_2)\right\|_2}{||x_1 - x_2||_2} \le K$$

.

Suppose $\left\|L^j - g\right\|_{Lip} < \epsilon$.
Then

$$\frac{\|g(x_1) - g(x_2)\|_2}{\|x_1 - x_2\|_2} \geq \frac{\left\|L^j(x_1) - L^j(x_2)\right\|_2}{\|x_1 - x_2\|_2} - \left\|L^j - g\right\|_{Lip} \geq K' - \epsilon$$

.

Therefore if $\epsilon < K'$, $\frac{\|g(x_1)-g(x_2)\|_2}{\|x_1-x_2\|_2} \geq K'$ for $x_1 \neq x_2$, which means that g is invertible.

□

Note that the Lipschitz norm of a linear function is the largest eigenvalue of the matrix of the function. From below the Lipschitz ratio can be estimated by the smallest eigenvalue.

7.2 Injectivity Criterion

In this section we show that injectivity of the characteristic map can be inferred from sufficiently close linear approximation.

First, we note that Theorem 3.1 from [5] works also for *Haar subdivision* defined below.

Definition 7.1. *On a regular complex define Haar subdivision by*

$$\begin{aligned} p^{j+1}(v(v_{ij}, v_{i(j+1)}) &= \\ p^{j+1}(v(v_{ij}, v_{(i+1)j}) &= \\ p^{j+1}(v(v_{ij}, v_{(i+1)(j+1)}) &= \\ p^{j+1}(v_{ij}) &= p^j(v_{ij}) \end{aligned}$$

In the formulas above we used notation $v(u, w)$ for the vertex of $D(\mathcal{R})$ inserted at the edge (u, w).

Note that this subdivision process is not stationary in our sense and the limit function is discontinuous (piecewise-constant.) Convergence however, is obvious, and basis functions satisfy the usual relation. Explicitly, the basis functions are defined on domains $\mathcal{T}_L(v_{ij}) = \mathcal{T}((v_{ij}, v_{i(j+1)}, v_{(i+1),(j+1)}) \cup \mathcal{T}((v_ij, v_{(i+1)j}, v_{(i+1),(j+1)}) \setminus ([v_{i(j+1)}, v_{(i+1)(j+1)}] \cup [v_{i(j+1)}, v_{(i+1)(j+1)}])$

$\phi_{v_{ij}} = \begin{cases} 1, & t \in \mathcal{T}_L(v_{ij}) \\ 0, & \text{otherwise} \end{cases}$

scaling relation for the basis functions is trivial. Stability is obvious.

If the comparison scheme B in Theorem 3.1 is taken to be B, $\mathcal{S}$ still converges and the estimate for the convergence rate is the same as in [5].

Theorem 7.1. *Suppose $\mathcal{S}$ is a convergent subdivision scheme on a finite locally regular complex K and K_T satisfies (7.3). Then the linear approximations converge to S in the Lipschitz norm, if S is C^1 on the regular complex.*

Proof. Consider convergence on a subcomplex of the regular complex. For the regular complex we introduce the difference scheme, following Dyn and Levin [8]:

$$(d^1 p)(v_{ij}) = 2^k (p^k_{(i+1)j} - P^k_{ij})$$

If $\mathcal{S}$ is C^1 there is a matrix subdivision scheme $\mathcal{S}^{(1)}$ such that

$$d^1 p^{k+1} = S^{(1)} dp^k$$

is convergent.

By a straightforward generalization of the Lemma 3.2 to the matrix case, piecewise-constant approximations of $d^1 p^k$ converge uniformly to $\nabla f[p^0]$ - the gradient of the limit function.

Note that by construction $d^1 p^k$ is the gradient of the piecewise linear approximation on each triangle of K_T. Consider two adjacent triangles of K. Let μ be the piecewise continuous affine map that maps them to triangles of a regular complex. The limit function $f \circ \mu^{-1}$ is continuous and piecewise C^1, with limits of gradient existing on the boundary. The difference scheme converges everywhere inside each triangle. On the boundary the limits on different sides may be different.

The linear approximation to the derivative in the direction of coordinate vector e_1 on each triangle of K is given by

$$2^k |\mu^{-1}(e_1)| (p^k_{(i+1)j} - p^k_{ij})$$

The same applies to e_2. Consider $\nabla(f[p] - L^j)$ on each triangle. It is equal to $J(\nabla(f \circ \mu^{-1}) - d^1 p^j)$ where J is the Jacobian matrix of μ on the triangle. Clearly, $\left\| f - L^j \right\|_2$ converges to 0 uniformly on K, as $\left\| \nabla(f \circ \mu^{-1}) - d^1 p^j \right\|_2$ converges to 0.

The conditions of Lemma 7.3 are satisfied for K_T, therefore, $\|f - L_j\|_{Lip} \leq B \left\| \|\nabla(f - L_j)\|_2 \right\|_\infty$ and $\|f - L_j\|_{Lip}$ also converges to 0. □

Now we observe that each layer defined in Section 4.1 is a complex satisfying the requirements of Theorem 7.1.

Provided that the norm of the gradients of the linear approximations of Φ are uniformly bounded from below by a positive number, the characteristic map is injective on each layer, and Reif's argument in Theorem 3.6 [19] applies in unchanged form. We conclude that the following theorem holds:

Theorem 7.2. *Consider a subdivision scheme $\mathcal{S}$, which is C^1 in the regular case. Suppose for a subdivision level j the linear approximation of the characteristic map Φ for some valence k is within ϵ of the limit function in L_∞ norm and the gradients of each component of the linear approximation are within ϵ of the gradients of the components of the characteristic map where they are defined. If there is a lower bound for the smallest eigenvalue of the linear approximation, which is greater then ϵ, and L^j is injective, then Φ is injective.*

This theorem allows us to use the following procedure for asserting smoothness of a subdivision scheme on the k-regular complex. First we estimate the rate of convergence of the scheme and the derivative scheme in the regular case. This allows us to estimate the number of subdivision steps which are required to guarantee that the linear approximation is within some given tolerance ϵ from the characteristic map and its directional derivatives within ϵ from the derivatives of the characteristic map. Then if the smaller eigenvalue of each linear piece of the approximation can be estimated from below by a positive number with tolerance ϵ, then the characteristic map is injective and subdivision is smooth.

8 Conclusions and Future Work

In this paper our main goal was to reduce analysis of subdivision around extraordinary vertices to analysis of subdivision on regular grids for particular configurations of initial values.

The criteria presented in this paper provide us with a sufficiently complete reduction for closed surfaces, except for the schemes that do not satisfy non-degeneracy and stability conditions.

Similar reduction with minor alterations can be done for surfaces with boundary.

The injectivity criterion in the last section can be used to check smoothness of subdivision schemes without having an explicit expression for the limit surface. However, such check has to be done for each valence k. it would be useful to obtain criteria for smoothness either directly in terms of the scheme coefficients, or at least in a form that would allow to check smoothness for all $k > k_0$ for some k_0.

Criteria for smoothness for non-equal eigenvalues have a number of degrees of freedom. It might be possible to use these degrees of freedom to obtain schemes which have tension parameters.

The degree estimates of Section 5.6 suggests that "non-flat" smoothness of order higher then 1 requires large supports. It would be interesting to find a sufficiently tight lower bound on the support of C^k schemes.

Perceptual smoothness of a surface can vary continuously, while the mathematical concept of C^k-continuity is discrete. A more natural measure of smoothness would be the Hölder exponent. Estimates for the Hölder exponents of the eigenbasis functions would be a good indication of "how close" a C^1 surface is to C^2, etc.

An important class of schemes is the class of symmetric schemes, which include most of the known schemes. More specific conditions for smoothness will be derived for such schemes in a subsequent paper.

Using conditions in the paper one can describe a number of general constructions of everywhere C^k subdivision schemes.

The connection to singularity theory is also worth exploring: singularity theory provides powerful tools for studies of functions depending on parameters. A subdivision surface can be considered as a function on $\mathcal{T}(U_1)$ with subdivision coefficients as parameters.

Finally, this study of stationary locally-defined subdivision provides a basis for extension to the non-local and non-stationary cases, which may allow us to produce surfaces of higher smoothness with smaller top-level support.

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